Math olympiad for the French Speaking, 2021
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–	Juniors
1	Let R and S be the numbers defined by $$R = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{223}{224} \text{ and } S = \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \cdots \times \frac{224}{225}.$$ Prove that $R < \frac{1}{15} < S$.
2	Evariste has drawn twelve triangles as follows, so that two consecutive triangles share exactly one edge. `https://cdn.artofproblemsolving.com/attachments/6/2/50377e7ad5fb1c40e36725e43c7eeb1e3c284 png` Sophie colors every triangle side in red, green or blue. Among the 3^{24} possible colorings, how many have the property that every triangle has one edge of each color?
3	Every point in the plane was colored in red or blue. Prove that one the two following statements is true: • there exist two red points at distance 1 from each other; • there exist four blue points B_1, B_2, B_3, B_4 such that the points B_i and B_j are at distance $\|i-j\|$ from each other, for all integers i and j such as $1 \leq i \leq 4$ and $1 \leq j \leq 4$.
4	Let $\mathbb{N}_{\geqslant 1}$ be the set of positive integers. Find all functions $f\colon \mathbb{N}_{\geqslant 1} \to \mathbb{N}_{\geqslant 1}$ such that, for all positive integers m and n: $$\mathrm{GCD}\left(f(m), n\right) + \mathrm{LCM}\left(m, f(n)\right) = \mathrm{GCD}\left(m, f(n)\right) + \mathrm{LCM}\left(f(m), n\right).$$ Note: if a and b are positive integers, $\mathrm{GCD}(a, b)$ is the largest positive integer that divides both a and b, and $\mathrm{LCM}(a, b)$ is the smallest positive integer that is a multiple of both a and b.
–	Seniors
1	Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be positive integers such that $a_{n+2} = a_n + a_{n+1}$ and $b_{n+2} = b_n + b_{n+1}$ for all $n \geq 1$. Assume that a_n divides b_n for infinitely many values of n. Prove that there exists an integer c such that $b_n = ca_n$ for all $n \geq 1$.
2	Albert and Beatrice play a game. 2021 stones lie on a table. Starting with Albert, they alternatively remove stones from the table, while obeying the following rule. At the n-th turn, the active player

(Albert if n is odd, Beatrice if n is even) can remove from 1 to n stones. Thus, Albert first removes 1 stone; then, Beatrice can remove 1 or 2 stones, as she wishes; then, Albert can remove from 1 to 3 stones, and so on.
The player who removes the last stone on the table loses, and the other one wins. Which player has a strategy to win regardless of the other player's moves?

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3 Let $ABCD$ be a square with incircle Γ. Let M be the midpoint of the segment $[CD]$. Let $P \neq B$ be a point on the segment $[AB]$. Let $E \neq M$ be the point on Γ such that (DP) and (EM) are parallel. The lines (CP) and (AD) meet each other at F. Prove that the line (EF) is tangent to Γ

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4 Let $\mathbb{N}_{\geq 1}$ be the set of positive integers.
Find all functions $f\colon \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$ such that, for all positive integers m and n:

(a) $n = (f(2n) - f(n))\,(2f(n) - f(2n))$,
(b)$f(m)f(n) - f(mn) = (f(2m) - f(m))\,(2f(n) - f(2n)) + (f(2n) - f(n))\,(2f(m) - f(2m))$,
(c) $m - n$ divides $f(2m) - f(2n)$ if m and n are distinct odd prime numbers.
