USA Team Selection Test 2016

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by v_Enhance

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1 Let $S = \{1, \dots, n\}$. Given a bijection $f : S \to S$ an *orbit* of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections f_1, ..., f_k from S to itself, prove that

$$c(f_1) + \cdots + c(f_k) \le n(k-1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

Proposed by Maria Monks Gillespie

2 Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

Proposed by Evan Chen

3 Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

Here, a polynomial Q divides P if there exists $R \in \mathbb{F}_p[x]$ such that $P(x) - Q(x)R(x)$ is the polynomial with all coefficients 0 (with all addition and multiplication in the coefficients taken modulo p), and the gcd of two polynomials is the highest degree polynomial with leading coefficient 1 which divides both of them. A non-zero polynomial is a polynomial with not all coefficients 0. As an example of multiplication, $(x+1)(x+2)(x+3) = x^3 + x^2 + x + 1$ in $\mathbb{F}_5[x]$.

Proposed by Mark Sellke

Jan January 21st, 2016

1 Let $\sqrt{3} = 1.b_1b_2b_3\ldots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits b_n, b_{n+1}, $\ldots$, b_{2n} equals 1.

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2 Let $n \ge 4$ be an integer. Find all functions $W : \{1, \ldots, n\}^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a\in A}\sum_{b\in B}\sum_{c\in C} W(a,b)W(b,c) = |A||B||C|.$$

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3 Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.
