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– 1st Grade

Problem 1 Find all real solutions of the system of equations

$$x^2 - y^2 = 2(xz + yz + x + y),$$
$$y^2 - z^2 = 2(yx + zx + y + z),$$
$$z^2 - x^2 = 2(zy + xy + z + x).$$

Problem 2 Prove that the medians from the vertices A and B of a triangle ABC are orthogonal if and only if $BC^2 + AC^2 = 5AB^2$.

Problem 3 Prove that for any three real numbers x, y, z the following inequality holds:

$$|x| + |y| + |z| - |x + y| - |y + z| - |z + x| + |x + y + z| \geq 0.$$

Problem 4 The sequence $1, 2, 3, 4, 0, 9, 6, 9, 4, 8, 7, \ldots$ is formed so that each term, starting from the fifth, is the units digit of the sum of the previous four.
(a) Do the digits $2, 0, 0, 4$ occur in the sequence in this order?
(b) Will the initial digits $1, 2, 3, 4$ ever occur again in this order?

– 2nd Grade

Problem 1 Parts of a pentagon have areas x, y, z as shown in the picture. Given the area x, find the areas y and z and the area of the entire pentagon.
`https://services.artofproblemsolving.com/download.php?id=YXR0YWNobWVudHMvOS9mLzM5NjNjNDcw`
`=\&rn=U2NyZWVuIFNob3QgMjAyMS0wNC0wOCBhdCA0LjMwLjU1IFBNLnBuZw`

Problem 2 If a, b, c are positive numbers, prove the inequality

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}.$$

Problem 3 The sequence $(p_n)_{n\in\mathbb{N}}$ is defined by $p_1 = 2$ and, for $n \geq 2$, p_n is the largest prime factor of $p_1p_2\cdots p_{n-1} + 1$. Show that $p_n \neq 5$ for all n.

Problem 4 A frog jumps on the coordinate lattice, starting from the point $(1, 1)$, according to the following rules:

(i) From point (a, b) the frog can jump to either $(2a, b)$ or $(a, 2b)$;
(ii) If $a > b$, the frog can also jump from (a, b) to $(a - b, b)$, while for $a < b$ it can jump from (a, b) to $(a, b - a)$.
Can the frog get to the point: (a) $(24, 40)$; (b) $(40, 60)$; (c) $(24, 60)$; (d) $(200, 4)$?

– 3rd Grade

Problem 1 Let $ABCD$ be a square and P be a point on the shorter arc AB of the circumcircle of the square. Which values can the expression $\frac{AP+BP}{CP+DP}$ take?

Problem 2 If a, b, c are the sides and α, β, γ the corresponding angles of a triangle, prove the inequality

$$\frac{\cos\alpha}{a^3} + \frac{\cos\beta}{b^3} + \frac{\cos\gamma}{c^3} \geq \frac{3}{2abc}.$$

Problem 3 The altitudes of a tetrahedron meet at a single point. Prove that this point, the centroid of one face of the tetrahedron, the foot of the altitude on that face, and the three points dividing the other three altitudes in ratio $2 : 1$ (closer to the feet) all lie on a sphere.

Problem 4 Finitely many cells of an infinite square board are colored black. Prove that one can choose finitely many squares in the plane of the board so that the following conditions are satisfied:

(i) The interiors of any two different squares are disjoint;
(ii) Each black cell lies in one of these squares;
(iii) In each of these squares, the black cells cover at least $\frac{1}{5}$ and at most $\frac{4}{5}$ of the area of that square.

– 4th Grade

Problem 1 Let $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ $(n \in \mathbb{N})$ be complex numbers such that

$$|\epsilon_1 z_1 + \ldots + \epsilon_n z_n| \leq |\epsilon_1 w_1 + \ldots + \epsilon_n w_n|$$

holds for every choice of $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$. Prove that

$$|z_1|^2 + \ldots + |z_n|^2 \leq |w_1|^2 + \ldots + |w_n|^2.$$

Problem 2 Points P and Q inside a triangle ABC with sides a, b, c and the corresponding angle α, β, γ satisfy $\angle BPC = \angle CPA = \angle APB = 120^\circ$ and $\angle BQC = 60^\circ + \alpha$, $\angle CQA = 60^\circ + \beta$, $\angle AQB = 60^\circ + \gamma$. Prove the equality

$$(AP + BP + CP)^3 \cdot AQ \cdot BQ \cdot CQ = (abc)^2.$$

Problem 3 The sequences $(x_n), (y_n), (z_n), n \in \mathbb{N}$, are defined by the relations

$$x_{n+1} = \frac{2x_n}{x_n^2 - 1}, \qquad y_{n+1} = \frac{2y_n}{y_n^2 - 1}, \qquad z_{n+1} = \frac{2z_n}{z_n^2 - 1},$$

where $x_1 = 2$, $y_1 = 4$, and $x_1 y_1 z_1 = x_1 + y_1 + z_1$.
(a) Show that $x_n^2 \neq 1$, $y_n^2 \neq 1$, $z_n^2 \neq 1$ for all n;
(b) Does there exist a $k \in \mathbb{N}$ for which $x_k + y_k + z_k = 0$?

Problem 4 Determine all real numbers α with the property that all numbers in the sequence $\cos\alpha, \cos 2\alpha, \cos 2^2\alpha, \ldots$ are negative.
