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– 1st Grade

Problem 1 Show that a triangle whose side lengths are prime numbers cannot have integer area.

Problem 2 The product of the positive real numbers x, y, z is 1. Show that if

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq x+y+z$$

then

$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \geq x^k+y^k+z^k$$

for all positive integers k.

Problem 3 In an isosceles triangle with base a, lateral side b, and height to the base v, it holds that $\frac{a}{2}+v \geq b\sqrt{2}$. Find the angles of the triangle. Compute its area if $b=8\sqrt{2}$.

Problem 4 How many divisors of 30^{2003} are there which do not divide 20^{2000}?

– 2nd Grade

Problem 1 Find all pairs of real numbers (x, y) satisfying

$$(2x+1)^2+y^2+(y-2x)^2=\frac{1}{3}.$$

Problem 2 Let M be a point inside square $ABCD$ and A_1, B_1, C_1, D_1 be the second intersection points of AM, BM, CM, DM with the circumcircle of the square. Prove that $A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1$.

Problem 3 For positive numbers $a_1, a_2, \ldots, a_n$ ($n \geq 2$) denote $s = a_1 + \ldots + a_n$. Prove that

$$\frac{a_1}{s-a_1}+\ldots+\frac{a_n}{s-a_n} \geq \frac{n}{n-1}.$$

Problem 4 Find the least possible cardinality of a set A of natural numbers, the smallest and greatest of which are 1 and 100, and having the property that every element of A except for 1 equals the sum of two elements of A.

– 3rd Grade

Problem 1 Let a, b, c be the sides of triangle ABC and let α, β, γ be the corresponding angles.

(a) If $\alpha = 3\beta$, prove that $\left(a^2 - b^2\right)(a - b) = bc^2$.
(b) Is the converse true?

Problem 2 For every integer $n > 2$, prove the equality

$$\left\lfloor \frac{n(n+1)}{4n-2} \right\rfloor = \left\lfloor \frac{n+1}{4} \right\rfloor.$$

Problem 3 In a tetrahedron $ABCD$, all angles at vertex D are equal to α and all dihedral angles between faces having D as a vertex are equal to ϕ. Prove that there exists a unique α for which $\phi = 2\alpha$.

Problem 4 Given 8 unit cubes, 24 of their faces are painted in blue and the remaining 24 faces in red. Show that it is always possible to assemble these cubes into a cube of edge 2 on whose surface there are equally many blue and red unit squares.

– 4th Grade

Problem 1 Let I be a point on the bisector of angle BAC of a triangle ABC. Points M, N are taken on the respective sides AB and AC so that $\angle ABI = \angle NIC$ and $\angle ACI = \angle MIB$. Show that I is the incenter of triangle ABC if and only if points M, N and I are collinear.

Problem 2 A sequence $(a_n)_{n\geq 0}$ satisfies $a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right)$ for all integers m, n with $m \geq n \geq 0$. Given that $a_1 = 1$, find a_{2003}.

Problem 3 The natural numbers 1 through 2003 are arranged in a sequence. We repeatedly perform the following operation: If the first number in the sequence is k, the order of the first k terms is reversed. Prove that after several operations number 1 will occur on the first place.

Problem 4 Prove that the number $\binom{n}{p} - \left\lfloor \frac{n}{p} \right\rfloor$ is divisible by p for every prime number and integer $n \geq p$.