www.artofproblemsolving.com/community/c1983190
by Math_fj

– Category: Secondary

1 How many ordered pairs of integers (m, n) are there such that m and n are the legs of a right triangle with an area equal to a prime number not exceeding 80?

2 Let x and y be positive integers such that $2(x+y) = gcd(x,y) + lcm(x,y)$. Find $\frac{lcm(x,y)}{gcd(x,y)}$.

3 Let r be a positive real number. Denote by $[r]$ the integer part of r and by $\{r\}$ the fractional part of r. For example, if $r = 32.86$, then $\{r\} = 0.86$ and $[r] = 32$. What is the sum of all positive numbers r satisfying $25\{r\} + [r] = 125$?

4 $ABCD$ is an isosceles trapezium such that $AD = BC$, $AB = 5$ and $CD = 10$. A point E on the plane is such that $AE \perp EC$ and $BC = EC$. The length of AE can be expressed as $a\sqrt{b}$, where a and b are integers and b is not divisible by any square number other than 1. Find the value of $(a+b)$.

5 $g(x) : \mathbb{Z} \to \mathbb{Z}$ is a function that satisfies

$$g(x) + g(y) = g(x+y) - xy.$$

If $g(23) = 0$, what is the sum of all possible values of $g(35)$?

6 On a table near the sea, there are N glass boxes where $N < 2021$, each containing exactly 2021 balls. Sowdha and Rafi play a game by taking turns on the boxes where Sowdha takes the first turn. In each turn, a player selects a non-empty box and throws out some of the balls from it into the sea. If a player wants, he can throw out all of the balls in the selected box. The player who throws out the last ball wins. Let S be the sum of all values of N for which Sowdha has a winning strategy and let R be the sum of all values of N for which Rafi has a winning strategy. What is the value of $\frac{R-S}{10}$?

7 For a positive integer n, let $s(n)$ and $c(n)$ be the number of divisors of n that are perfect squares and perfect cubes respectively. A positive integer n is called fair if $s(n) = c(n) > 1$. Find the number of fair integers less than 100.

8 Let ABC be an acute-angled triangle. The external bisector of $\angle BAC$ meets the line BC at point N. Let M be the midpoint of BC. P and Q are two points on line AN such that, $\angle PMN =$

$\angle MQN = 90°$. If $PN = 5$ and $BC = 3$, then the length of QA can be expressed as $\frac{a}{b}$ where a and b are coprime positive integers. What is the value of $(a+b)$?

---

9 Cynthia loves Pokemon and she wants to catch them all. In Victory Road, there are a total of 80 Pokemon. Cynthia wants to catch as many of them as possible. However, she cannot catch any two Pokemon that are enemies with each other. After exploring around for a while, she makes the following two observations:
1. Every Pokemon in Victory Road is enemies with exactly two other Pokemon.
2. Due to her inability to catch Pokemon that are enemies with one another, the maximum number of the Pokemon she can catch is equal to n.
What is the sum of all possible values of n?

---

10 A positive integer n is called *nice* if it has at least 3 proper divisors and it is equal to the sum of its three largest proper divisors. For example, 6 is *nice* because its largest three proper divisors are $3, 2, 1$ and $6 = 3 + 2 + 1$. Find the number of *nice* integers not greater than 3000.

---

11 Let $ABCD$ be a square such that $A = (0, 0)$ and $B = (1, 1)$. $P(\frac{2}{7}, \frac{1}{4})$ is a point inside the square. An ant starts walking from P, touches 3 sides of the square and comes back to the point P. The least possible distance traveled by the ant can be expressed as $\frac{\sqrt{a}}{b}$, where a and b are integers and a not divisible by any square number other than 1. What is the value of $(a + b)$?

---

12 Two toads named Gamakichi and Gamatatsu are sitting at the points $(0, 0)$ and $(2, 0)$ respectively. Their goal is to reach $(5, 5)$ and $(7, 5)$ respectively by making one unit jumps in positive x or y direction at a time. How many ways can they do this while ensuring that there is no point on the plane where both Gamakichi And Gamatatsu land on?
