Bundeswettbewerb Mathematik 2021

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–	Round 1
1	A cube with side length 10 is divided into two cuboids with integral side lengths by a straight cut. Afterwards, one of these two cuboids is divided into two cuboids with integral side lengths by another straight cut. What is the smallest possible volume of the largest of the three cuboids?
2	The fraction $\frac{3}{10}$ can be written as a sum of two reciprocals in exactly two ways: $$\frac{3}{10}=\frac{1}{5}+\frac{1}{10}=\frac{1}{4}+\frac{1}{20}$$ a) In how many ways can $\frac{3}{2021}$ be written as a sum of two reciprocals? b) Is there a positive integer n not divisible by 3 with the property that $\frac{3}{n}$ can be written as a sum of two reciprocals in exactly 2021 ways?
3	Consider a triangle ABC with $\angle ACB=120^\circ$. Let A',B',C' be the points of intersection of the angular bisector through A, B and C with the opposite side, respectively. Determine $\angle A'C'B'$.
4	Consider a pyramid with a regular n-gon as its base. We colour all the segments connecting two of the vertices of the pyramid except for the sides of the base either red or blue. Show that if $n=9$ then for each such colouring there are three vertices of the pyramid connecting by three segments of the same colour, and that this is not necessarily the case if $n=8$.
–	Round 2
1	Let $Q(n)$ denote the sum of the digits of n in its decimal representation. Prove that for every positive integer k, there exists a multiple n of k such that $Q(n)=Q(n^2)$.
2	A school has 2021 students, each of which knows at least 45 of the other students (where ”knowing” is mutual). Show that there are four students who can be seated at a round table such that each of them knows both of her neighbours.
3	We are given a circle k and a point A outside of k. Next we draw three lines through A: one secant intersecting the circle k at points B and C, and two tangents touching the circle$k$ at points D and E. Let F be the midpoint of DE.

Show that the line DE bisects the angle $\angle BFC$.

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4 In the Cartesian plane, a line segment is called *tame* if it lies parallel to one of the coordinate axes and its distance to this axis is an integer. Otherwise it is called *wild*.

Let m and n be odd positive integers. The rectangle with vertices $(0,0), (m,0), (m,n)$ and $(0,n)$ is partitioned into finitely many triangles. Let M be the set of these triangles. Assume that

(1) Each triangle from M has at least one tame side.
(2) For each tame side of a triangle from M, the corresponding altitude has length 1.
(3) Each wild side of a triangle from M is a common side of exactly two triangles from M.

Show that at least two triangles from M have two tame sides each.
