EGMO 2021

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–	Day 1
1	The number 2021 is fantabulous. For any positive integer m, if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?
2	Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that the equation $$f(xf(x) + y) = f(y) + x^2$$ holds for all rational numbers x and y. Here, $\mathbb{Q}$ denotes the set of rational numbers.
3	Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.
–	Day 2
4	Let ABC be a triangle with incenter I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A across the line EF lies on the line BC.
5	A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that - no three points in P lie on a line and - no two points in P lie on a line through the origin. A triangle with vertices in P is *fat* if O is strictly inside the triangle. Find the maximum number of fat triangles.
6	Does there exist a nonnegative integer a for which the equation $$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$ has more than one million different solutions (m, n) where m and n are positive integers?

[i]The expression $\lfloor x \rfloor$ denotes the integer part (or floor) of the real number x. Thus $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor \pi \rfloor = \lfloor 22/7 \rfloor = 3$, $\lfloor 42 \rfloor = 42$, and $\lfloor 0 \rfloor = 0$.[/i]