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– **Day 1** April 13

1 Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for positive integers a and b,

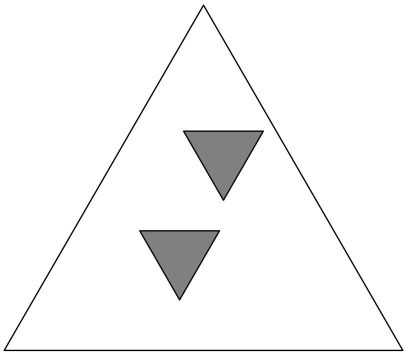
$$f(a^2 + b^2) = f(a)f(b) \text{ and } f(a^2) = f(a)^2.$$

2 Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

3 An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. (An example with $n = 2$ is drawn below.)



Prove that

$$n \leq \frac{2}{3}L^2.$$

– **Day 2** April 14

4 Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

(A lattice point is a point (x, y) in the coordinate plane where x and y are both integers, not necessarily positive.)

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5 A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying gcd$(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

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6 Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7 \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1 \end{aligned}$$
