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Art of Problem Solving

Iran Romanian Master of Mathematics TST 2021

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by Tintarn, Mr.C

– Day 1

1 Suppose that two circles α, β with centers P, Q, respectively , intersect orthogonally at A,B. Let CD be a diameter of β that is exterior to α. Let E, F be points on α such that CE, DF are tangent to α , with C, E on one side of PQ and D, F on the other side of PQ. Let S be the intersection of CF, AQ and T be the intersection of DE, QB. Prove that ST is parallel to CD and is tangent to α

2 Let $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying $f(x) = f(x+2) + 2f(x^2+2x)$. Prove that if for all $x > 1400^{2021}$, $xf(x) \leq 2021$, then $xf(x) \leq 2021$ for all $x \in \mathbb{R}^+$

Proposed by *Navid Safaei*

3 In a 3 by 3 table, by a k-worm, we mean a path of different cells $(S_1, S_2, ..., S_k)$ such that each two consecutive cells have one side in common. The k-worm at each steep can go one cell forward and turn to the $(S, S_1, ..., S_{k-1})$ if S is an unfilled cell which is adjacent (has one side in common) with S_1. Find the maximum number of k such that there is a k-worm $(S_1, ..., S_k)$ such that after finitly many steps can be turned to $(S_k, ..., S_1)$.

– Day 2

1 A polyomino is region with connected interior that is a union of a finite number of squares from a grid of unit squares. Do there exist a positive integer $n > 4$ and a polyomino P contained entirely within and n-by-n grid such that P contains exactly 3 unit squares in every row and every column of the grid?

Proposed by *Nikolai Beluhov*

2 Let ABC be a triangle with $AB \neq AC$ and with incenter I. Let M be the midpoint of BC, and let L be the midpoint of the circular arc BAC. Lines through M parallel to BI, CI meet AB, AC at E and F, respectively, and meet LB and LC at P and Q, respectively. Show that I lies on the radical axis of the circumcircles of triangles EMF and PMQ.

Proposed by *Andrew Wu*

3 We call a polynomial $P(x) = a_d x^d + ... + a_0$ of degree d *nice* if

$$\frac{2021(|a_d| + ... + |a_0|)}{2022} < max_{0 \leq i \leq d} |a_i|$$

Initially Shayan has a sequence of d distinct real numbers; $r_1, ..., r_d \neq \pm 1$. At each step he choose a positive integer $N > 1$ and raises the d numbers he has to the exponent of N, then delete the previous d numbers and constructs a monic polynomial of degree d with these number as roots, then examine whether it is nice or not. Prove that after some steps, all the polynomials that shayan produces would be nice polynomials

Proposed by *Navid Safaei*

– Day 3

1 Let $P(x) = x^{2016} + 2x^{2015} + ... + 2017, Q(x) = 1399x^{1398} + ... + 2x + 1$. Prove that there are strictly increasing sequances $a_i, b_i, i = 1, ...$ of positive integers such that $gcd(a_i, a_{i+1}) = 1$ for each i. Moreover, for each even i, $P(b_i) \nmid a_i, Q(b_i)|a_i$ and for each odd i, $P(b_i)|a_i, Q(b_i) \nmid a_i$

Proposed by *Shayan Talaei*

2 In a chess board we call a group of queens *independant* if no two are threatening each other. In an n by n grid, we put exaxctly one queen in each cell ofa greed. Let us denote by M_n the minimum number of independant groups that hteir union contains all the queens. Let k be a positive integer, prove that $M_{3k+1} \leq 3k + 2$

Proposed by *Alireza Haghi*

3 Let n be an integer greater than 1 such that n could be represented as a sum of the cubes of two rational numbers, prove that n is also the sum of the cubes of two non-negative rational numbers.

Proposed by *Navid Safaei*