The problems from the CCA Math Bonanza held on 4/17/2021.

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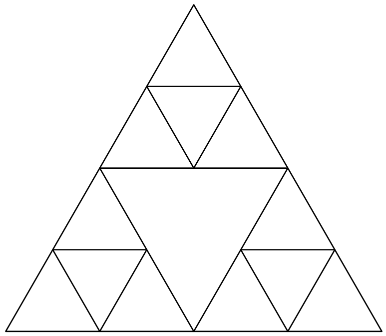
by mira74

–	Individual Round
I1	Compute the number of positive integer divisors of 2121 with a units digit of 1. *2021 CCA Math Bonanza Individual Round#1*
I2	Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Points P, Q, and R are chosen on segments BC, CA, and AB, respectively, such that triangles AQR, BPR, CPQ have the same perimeter, which is $\frac{4}{5}$ of the perimeter of PQR. What is the perimeter of PQR? *2021 CCA Math Bonanza Individual Round#2*
I3	How many reorderings of $2, 3, 4, 5, 6$ have the property that every pair of adjacent numbers are relatively prime? *2021 CCA Math Bonanza Individual Round#3*
I4	Given that nonzero real numbers x and y satisfy $x + \frac{1}{y} = 3$ and $y + \frac{1}{x} = 4$, what is $xy + \frac{1}{xy}$? *2021 CCA Math Bonanza Individual Round#4*
I5	If digits A, B, and C (between 0 and 9 inclusive) satisfy $\begin{array}{r} C\ C\ A \\ +\ B\ 2\ B \\ \hline A\ 8\ 8 \end{array}$ what is $A \cdot B \cdot C$? *2021 CCA Math Bonanza Individual Round#5*
I6	Let ABC be a right triangle with $AB = 3$, $BC = 4$, and $\angle B = 90^\circ$. Points P, Q, and R are chosen on segments AB, BC, and CA, respectively, such that PQR is an equilateral triangle, and $BP = BQ$. Given that BP can be written as $\frac{\sqrt{a}-b}{c}$, where a, b, c are positive integers and $\gcd(b, c) = 1$, what is $a + b + c$? *2021 CCA Math Bonanza Individual Round#6*

I7 The image below consists of a large triangle divided into 13 smaller triangles. Let N be the number of ways to color each smaller triangle one of red, green, and blue such that if T_1 and T_2 are smaller triangles whose perimeters intersect at more than one point, T_1 and T_2 have two different colors. Compute the number of positive integer divisors of N.



2021 CCA Math Bonanza Individual Round#7

I8 Joel is rolling a 6-sided die. After his first roll, he can choose to re-roll the die up to 2 more times. If he rerolls strategically to maximize the expected value of the final value the die lands on, the expected value of the final value the die lands on can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

2021 CCA Math Bonanza Individual Round#8

I9 Points A, B, C, D, and E are on the same plane such that A, E, C lie on a line in that order, B, E, D lie on a line in that order, $AE = 1$, $BE = 4$, $CE = 3$, $DE = 2$, and $\angle AEB = 60^\circ$. Let AB and CD intersect at P. The square of the area of quadrilateral $PAED$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

2021 CCA Math Bonanza Individual Round#9

I10 Let $\text{s}_b(k)$ denote the sum of the digits of k in base b. Compute

$$\text{s}_{101}(33) + \text{s}_{101}(66) + \text{s}_{101}(99) + \cdots + \text{s}_{101}(3333).$$

2021 CCA Math Bonanza Individual Round#10

I11 An triangle with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) has centroid at $(1, 1)$. The ratio between the lengths of the sides of the triangle is $3:4:5$. Given that

$$x_1^3 + x_2^3 + x_3^3 = 3x_1x_2x_3 + 20 \quad \text{and} \quad y_1^3 + y_2^3 + y_3^3 = 3y_1y_2y_3 + 21,$$

the area of the triangle can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

2021 CCA Math Bonanza Individual Round#11

I12 Let ABC be a triangle, let the A-altitude meet BC at D, let the B-altitude meet AC at E, and let $T \neq A$ be the point on the circumcircle of ABC such that $AT||BC$. Given that D, E, T are collinear, if $BD=3$ and $AD=4$, then the area of ABC can be written as $a+\sqrt{b}$, where a and b are positive integers. What is $a+b$?

2021 CCA Math Bonanza Individual Round#12

I13 Find the sum of the two smallest odd primes p such that for some integers a and b, p does not divide b, b is even, and $p^2=a^3+b^2$.

2021 CCA Math Bonanza Individual Round#13

I14 For an ordered 10-tuple of nonnegative integers $a_1, a_2, \ldots, a_{10}$, we denote

$$f(a_1, a_2, \ldots, a_{10}) = \left(\prod_{i=1}^{10} \binom{20-(a_1+a_2+\cdots+a_{i-1})}{a_i}\right) \cdot \left(\sum_{i=1}^{10} \binom{18+i}{19} a_i\right).$$

When $i=1$, we take $a_1+a_2+\cdots+a_{i-1}$ to be 0. Let N be the average of $f(a_1, a_2, \ldots, a_{10})$ over all 10-tuples of nonnegative integers $a_1, a_2, \ldots, a_{10}$ satisfying

$$a_1+a_2+\cdots+a_{10}=20.$$

Compute the number of positive integer divisors of N.

2021 CCA Math Bonanza Individual Round#14

I15 Let N be the number of functions f from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, \ldots, 255\}$ with the property that:

- $f(k)=1$ for some $k \in \{1,2,3,4,5,6,7,8\}$
- If $f(a)=f(b)$, then $a=b$.
- For all $n \in \{1,2,3,4,5,6,7,8\}$, if $f(n) \neq 1$, then $f(k)+1 > \frac{f(n)}{2} \geq f(k)$ for some $k \in \{1,2,\ldots,7,8\}$.
- For all $k, n \in \{1,2,3,4,5,6,7,8\}$, if $f(n)=2f(k)+1$, then $k<n$.

Compute the number of positive integer divisors of N.

2021 CCA Math Bonanza Individual Round#15

– Team Round

T1 How many sequences of words (not necessarily grammatically correct) have the property that the first word has one letter, each word can be obtained by inserting a letter somewhere in the previous word, and the final word is CCAMT? Here are examples of possible sequences:

C,CA,CAM,CCAM,CCAMT.

A,AT,CAT,CAMT,CCAMT.

2021 CCA Math Bonanza Team Round#1

T2 Given that real numbers a, b, and c satisfy $ab = 3$, $ac = 4$, and $b + c = 5$, the value of bc can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

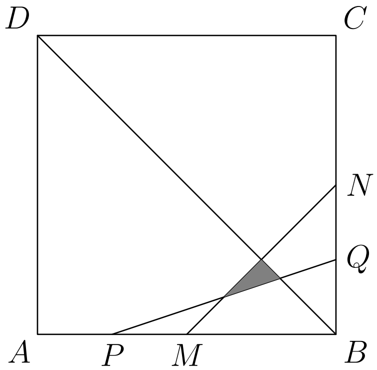
2021 CCA Math Bonanza Team Round#2

T3 For any real number x, we let $\lfloor x \rfloor$ be the unique integer n such that $n \leq x < n + 1$. For example. $\lfloor 31.415 \rfloor = 31$. Compute

$$2020^{2021} - \left\lfloor \frac{2020^{2021}}{2021} \right\rfloor (2021).$$

2021 CCA Math Bonanza Team Round#3

T4 Let $ABCD$ be a unit square. Points M and N are the midpoints of sides AB and BC respectively. Let P and Q be the midpoints of line segments AM and BN respectively. Find the reciprocal of the area of the triangle enclosed by the three line segments PQ, MN, and DB.



2021 CCA Math Bonanza Team Round#4

T5 We say that a *special word* is any sequence of letters **containing a vowel**. How many ordered triples of special words (W_1, W_2, W_3) have the property that if you concatenate the three words, you obtain a rearrangement of ”aadvarks”?

For example, the number of triples of special words such that the concatenation is a rearrangement of “adaa” is 6, and all of the possible triples are:

(da,a,a),(ad,a,a),(a,da,a),(a,ad,a),(a,a,da),(a,a,ad).

2021 CCA Math Bonanza Team Round#5

T6 Three spheres have radii 144, 225, and 400, are pairwise externally tangent to each other, and are all tangent to the same plane at A, B, and C. Compute the area of triangle ABC.

2021 CCA Math Bonanza Team Round#6

T7 Find the sum of all positive integers n with the following properties:

- n is not divisible by any primes larger than 10.
- For some positive integer k, the positive divisors of n are

$$1 = d_1 < d_2 < d_3 \cdots < d_{2k} = n.$$

- The divisors of n have the property that

$$d_1 + d_2 + \cdots + d_k = 3k.$$

2021 CCA Math Bonanza Team Round#7

T8 Let ABC be a triangle with $AB = 9$ and $AC = 12$. Point B' is chosen on line AC such that the midpoint of B and B' is equidistant from A and C. Point C' is chosen similarly. Given that the circumcircle of $AB'C'$ is tangent to BC, compute BC^2.

2021 CCA Math Bonanza Team Round#8

T9 Each number in the list $1, 2, 3, \ldots, 10$ is either colored red or blue. Numbers are colored independently, and both colors are equally probable. The expected value of the number of positive integers expressible as a sum of a red integer and a blue integer can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

2021 CCA Math Bonanza Team Round#9

T10 Given that positive integers a, b satisfy

$$\frac{1}{a+\sqrt{b}} = \sum_{i=0}^{\infty} \frac{\sin^2\left(\frac{10^\circ}{3^i}\right)}{\cos\left(\frac{30^\circ}{3^i}\right)},$$

where all angles are in degrees, compute $a + b$.

2021 CCA Math Bonanza Team Round#10

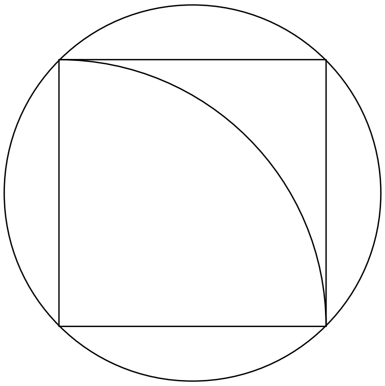
– Lightning Round

L1.1 Compute

$$(2+0\cdot 2\cdot 1)+(2+0-2)\cdot(1)+(2+0)\cdot(2-1)+(2)\cdot\left(0+2^{-1}\right).$$

2021 CCA Math Bonanza Lightning Round#1.1

L1.2 A square is inscribed in a circle of radius 6. A quarter circle is inscribed in the square, as shown in the diagram below. Given the area of the region inside the circle but outside the quarter circle is $n\pi$ for some positive integer n, what is n?



2021 CCA Math Bonanza Lightning Round#1.2

L1.3 A coin is flipped 20 times. Let p be the probability that each of the following sequences of flips occur exactly twice:

- one head, two tails, one head
- one head, one tails, two heads.

Given that p can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $\gcd(m, n)$.

2021 CCA Math Bonanza Lightning Round#1.3

L1.4 On Day 1, Alice starts with the number $a_1 = 5$. For all positive integers $n > 1$, on Day n, Alice randomly selects a positive integer a_n between a_{n-1} and $2a_{n-1}$, inclusive. Given that the probability that all of $a_2, a_3, \ldots, a_7$ are odd can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $m + n$.

2021 CCA Math Bonanza Lightning Round#1.4

L2.1 Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. The line through A perpendicular to AC intersects line BC at D, and the line through C perpendicular to AC intersects line AB at E. Compute the area of triangle BDE.

2021 CCA Math Bonanza Lightning Round#2.1

L2.2 Given that nonzero reals a, b, c, d satisfy $a^b = c^d$ and $\frac{a}{2c} = \frac{b}{d} = 2$, compute $\frac{1}{c}$.

2021 CCA Math Bonanza Lightning Round#2.2

L2.3 Broady The Boar is playing a boring board game consisting of a circle with 2021 points on it, labeled 0, 1, 2, ... 2020 in that order clockwise. Broady is rolling 2020-sided die which randomly produces a whole number between 1 and 2020, inclusive.

Broady starts at the point labelled 0. After each dice roll, Broady moves up the same number of points as the number rolled (point 2020 is followed by point 0). For example, if they are at 0 and roll a 5, they end up at 5. If they are at 2019 and roll a 3, they end up at 1.

Broady continues rolling until they return to the point labelled 0. What is the expected number of times they roll the dice?

2021 CCA Math Bonanza Lightning Round#2.3

L2.4 Compute the number of two digit positive integers that are divisible by both of their digits. For example, 36 is one of these two digit positive integers because it is divisible by both 3 and 6.

2021 CCA Math Bonanza Lightning Round#2.4

L3.1 A point is chosen uniformly at random from the interior of a unit square. Let p be the probability that any circle centered at the point that intersects a diagonal of the square must also intersect a side of the square. Given that p^2 can be written as $m-\sqrt{n}$ for positive integers m and n, what is $m+n$?

2021 CCA Math Bonanza Lightning Round#3.1

L3.2 A frog is standing in a center of a 3×3 grid of lilypads. Each minute, the frog chooses a square that shares exactly one side with their current square uniformly at random, and jumps onto the lilypad on their chosen square. The frog stops jumping once it reaches a lilypad on a corner of the grid. What is the expected number of times the frog jumps?

2021 CCA Math Bonanza Lightning Round#3.2

L3.3 Compute the smallest positive integer that gives a remainder of 1 when divided by 11, a remainder of 2 when divided by 21, and a remainder of 5 when divided by 51.

2021 CCA Math Bonanza Lightning Round#3.3

L3.4 Compute the sum of $x^2 + y^2$ over all four ordered pairs (x, y) of real numbers satisfying $x = y^2 - 20$ and $y = x^2 + x - 21$.

2021 CCA Math Bonanza Lightning Round#3.4

L4.1 Suppose that $x^2 + px + q$ has two distinct roots $x = a$ and $x = b$. Furthermore, suppose that the positive difference between the roots of $x^2 + ax + b$, the positive difference between the roots of $x^2 + bx + a$, and twice the positive difference between the roots of $x^2 + px + q$ are all equal. Given that q can be expressed in the form $\frac{m}{m}$, where m and n are relatively prime positive integers, compute $m + n$.

2021 CCA Math Bonanza Lightning Round#4.1

L4.2 Compute the number of (not necessarily convex) polygons in the coordinate plane with the following properties:

- If the coordinates of a vertex are (x, y), then x, y are integers and $1 \leq |x| + |y| \leq 3$
- Every side of the polygon is parallel to either the x or y axis
- The point $(0, 0)$ is contained in the interior of the polygon.

2021 CCA Math Bonanza Lightning Round#4.2

L4.3 For a positive integer n, let $f(n)$ be the sum of the positive integers that divide at least one of the nonzero base 10 digits of n. For example, $f(96) = 1 + 2 + 3 + 6 + 9 = 21$. Find the largest positive integer n such that for all positive integers k, there is some positive integer a such that $f^k(a) = n$, where $f^k(a)$ denotes f applied k times to a.

2021 CCA Math Bonanza Lightning Round#4.3

L4.4 Let ABC be a triangle with $AB = 5$, $BC = 7$, $CA = 8$, and let M be the midpoint of BC. Points P and Q are chosen on the circumcircle of ABC such that MPQ and ABC are similar (with vertices in that order). The product of all different possible areas of MPQ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

2021 CCA Math Bonanza Lightning Round#4.4

L5.1 Estimate the number of distinct submissions to this problem. Your submission must be a positive integer less than or equal to 50. If you submit E, and the actual number of distinct submissions is D, you will receive a score of $\frac{2}{0.5|E-D|+1}$.

2021 CCA Math Bonanza Lightning Round#5.1

L5.2 Define the sequences $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ such that $x_0 = 1$, $y_0 = 2021$, and for all nonnegative integers n, we have $x_{n+1} = \sqrt{x_n y_n}$ and $y_{n+1} = \frac{x_n + y_n}{2}$. There is some constant X such that as n grows large, $x_n - X$ and $y_n - X$ both approach 0. Estimate X.

An estimate of E earns $\max(0, 2 - 0.02|A - E|)$ points, where A is the actual answer.

2021 CCA Math Bonanza Lightning Round#5.2

L5.3 Let N be the number of sequences of words (not necessarily grammatically correct) that have the property that the first word has one letter, each word can be obtained by inserting a letter somewhere in the previous word, and the final word is CCAMATHBONANZA. Here is an example of a possible sequence:

N, NA, NZA, BNZA, BNAZA, BONAZA, BONANZA, CBONANZA, CABONANZA, CAMBONANZA, CAMABONANZA, CAMAHBONANZA, CCAMAHBONANZA, CCAMATHBONANZA.

Estimate $\frac{N}{12!}$. An estimate of $E > 0$ earns $\max(0, 4 - 2\max(A/E, E/A))$ points, where A is the actual answer. An estimate of $E = 0$ earns 0 points.

2021 CCA Math Bonanza Lightning Round#5.3

L5.4 Estimate the number of primes among the first thousand primes divide some term of the sequence

$$2^0 + 1, 2^1 + 1, 2^2 + 1, 2^3 + 1, \ldots.$$

An estimate of E earns $2^{1-0.02|A-E|}$ points, where A is the actual answer.

2021 CCA Math Bonanza Lightning Round#5.4

– Tiebreaker Round

TB1 Consider the set of all ordered 6-tuples of nonnegative integers (a, b, c, d, e, f) such that

$$a + 2b + 6c + 30d + 210e + 2310f = 2^{15}.$$

In the tuple with the property that $a + b + c + d + e + f$ is minimized, what is the value of c?

2021 CCA Math Bonanza Tiebreaker Round#1

TB2 Convex quadrilateral $ABCD$ with perpendicular diagonals satisfies $\angle B = \angle C = 90^\circ$, $BC = 20$, and $AD = 30$. Compute the square of the area of a triangle with side lengths equal to CD, DA, and AB.

2021 CCA Math Bonanza Tiebreaker Round#2

TB3 In a party of 2020 people, some pairs of people are friends. We say that a given person's *popularity* is the size of the largest group of people in the party containing them with the property that every pair of people in that group is friends. A person has popularity number 1 if they have no friends. What is the largest possible number of distinct popularities in the party?

2021 CCA Math Bonanza Tiebreaker Round#3

TB4 For all integers $0 \le k \le 16$, let

$$S_k = \sum_{j=0}^{k} (-1)^j \binom{16}{j}.$$

Compute $\max(S_0, S_1, \dots S_{16})$.

2021 CCA Math Bonanza Tiebreaker Round#4
