

All-Russian Olympiad 2021

www.artofproblemsolving.com/community/c1988301

by Tintarn, Kamran011, VicKmath7, Ya_pank

– Grade 9

-
- 1** On a circle there're 1000 marked points, each colored in one of k colors. It's known that among any 5 pairwise intersecting segments, endpoints of which are 10 distinct marked points, there're at least 3 segments, each of which has its endpoints colored in different colors. Determine the smallest possible value of k for which it's possible.
-
- 2** Let n be a natural number. An integer $a > 2$ is called n -decomposable, if $a^n - 2^n$ is divisible by all the numbers of the form $a^d + 2^d$, where $d \neq n$ is a natural divisor of n . Find all composite $n \in \mathbb{N}$, for which there's an n -decomposable number.
-
- 3** On a line $n + 1$ segments are marked such that one of the points of the line is contained in all of them. Prove that one can find 2 distinct segments I, J which intersect at a segment of length at least $\frac{n-1}{n}d$, where d is the length of the segment I .
-
- 4** Given an acute triangle ABC , point D is chosen on the side AB and a point E is chosen on the extension of BC beyond C . It became known that the line through E parallel to AB is tangent to the circumcircle of $\triangle ADC$. Prove that one of the tangents from E to the circumcircle of $\triangle BCD$ cuts the angle $\angle ABE$ in such a way that a triangle similar to $\triangle ABC$ is formed.
-
- 5** The reals $b > 0$ and a are such that the quadratic $x^2 + ax + b$ has two distinct real roots, exactly one of which lies in the interval $[-1; 1]$. Prove that one of the roots lies in the interval $(-b; b)$.
-
- 6** Given is a non-isosceles triangle ABC with $\angle ABC = 60^\circ$, and in its interior, a point T is selected such that $\angle ATC = \angle BTC = \angle BTA = 120^\circ$. Let M the intersection point of the medians in ABC . Let TM intersect (ATC) at K . Find TM/MK .
-
- 7** Given are positive integers $n > 20$ and $k > 1$, such that k^2 divides n . Prove that there exist positive integers a, b, c , such that $n = ab + bc + ca$.
-
- 8** One hundred sages play the following game. They are waiting in some fixed order in front of a room. The sages enter the room one after another. When a sage enters the room, the following happens - the guard in the room chooses two arbitrary distinct numbers from the set $1, 2, 3$, and announces them to the sage in the room. Then the sage chooses one of those numbers, tells it to the guard, and leaves the room, and the next enters, and so on. During the game, before a sage chooses a number, he can ask the guard what were the chosen numbers of the previous two sages. During the game, the sages cannot talk to each other. At the end, when everyone has

finished, the game is considered as a failure if the sum of the 100 chosen numbers is exactly 200; else it is successful. Prove that the sages can create a strategy, by which they can win the game.

– Grade 10

1 On the side BC of the parallelogram $ABCD$, points E and F are given (E lies between B and F) and the diagonals AC, BD meet at O . If it's known that AE, DF are tangent to the circumcircle of $\triangle AOD$, prove that they're tangent to the circumcircle of $\triangle EOF$ as well.

2 Find all sets of positive integers $\{x_1, x_2, \dots, x_{20}\}$ such that

$$x_{i+2}^2 = \text{lcm}(x_{i+1}, x_i) + \text{lcm}(x_i, x_{i-1})$$

for $i = 1, 2, \dots, 20$ where $x_0 = x_{20}, x_{21} = x_1, x_{22} = x_2$.

3 In the country there're N cities and some pairs of cities are connected by two-way airlines (each pair with no more than one). Every airline belongs to one of k companies. It turns out that it's possible to get to any city from any other, but it fails when we delete all airlines belonging to any one of the companies. What is the maximum possible number of airlines in the country?

4 Given a natural number $n > 4$ and $2n + 4$ cards numbered with $1, 2, \dots, 2n + 4$. On the card with number m a real number a_m is written such that $\lfloor a_m \rfloor = m$. Prove that it's possible to choose 4 cards in such a way that the sum of the numbers on the first two cards differs from the sum of the numbers on the two remaining cards by less than

$$\frac{1}{n - \sqrt{\frac{n}{2}}}$$

5 A teacher and her 30 students play a game on an infinite cell grid. The teacher starts first, then each of the 30 students makes a move, then the teacher and so on. On one move the person can color one unit segment on the grid. A segment cannot be colored twice. The teacher wins if, after the move of one of the 31 players, there is a 1×2 or 2×1 rectangle, such that each segment from it's border is colored, but the segment between the two adjacent squares isn't colored. Prove that the teacher can win.

6 Given is a polynomial $P(x)$ of degree $n > 1$ with real coefficients. The equation $P(P(P(x))) = P(x)$ has n^3 distinct real roots. Prove that these roots could be split into two groups with equal arithmetic mean.

7 Same as 9.7

- 8 Given is a cyclic pentagon $ABCDE$, inscribed in a circle k . The line CD intersects AB and AE in X and Y respectively. Segments EX and BY intersect again at P , and they intersect k in Q and R , respectively. Point A' is reflection of A across CD . The circles (PQR) and $(A'XY)$ intersect at M and N . Prove that CM and DN intersect on (PQR) .

– Grade 11

- 1 For some positive integer $n > m$, it turns out that it is representable as sum of 2021 non-negative integer powers of m , and that it is representable as sum of 2021 non-negative integer powers of $m + 1$. Find the maximal value of the positive integer m .
-

- 2 Let $P(x)$ be a nonzero polynomial of degree $n > 1$ with nonnegative coefficients such that function $y = P(x)$ is odd. Is that possible that for some pairwise distinct points A_1, A_2, \dots, A_n on the graph $G : y = P(x)$ the following conditions hold: tangent to G at A_1 passes through A_2 , tangent to G at A_2 passes through A_3, \dots , tangent to G at A_n passes through A_1 ?
-

- 3 Some language has only three letters - A, B and C . A sequence of letters is called a word iff it contains exactly 100 letters such that exactly 40 of them are consonants and other 60 letters are all A . What is the maximum numbers of words one can pick such that any two picked words have at least one position where they both have consonants, but different consonants?
-

- 4 In triangle ABC angle bisectors AA_1 and CC_1 intersect at I . Line through B parallel to AC intersects rays AA_1 and CC_1 at points A_2 and C_2 respectively. Let O_a and O_c be the circumcenters of triangles AC_1C_2 and CA_1A_2 respectively. Prove that $\angle O_aBO_c = \angle AIC$
-

5 Same as 10.5

- 6 In tetrahedron $ABCS$ no two edges have equal length. Point A' in plane BCS is symmetric to S with respect to the perpendicular bisector of BC . Points B' and C' are defined analogously. Prove that planes $ABC, AB'C', A'BC'$ and $A'B'C$ share a common point.
-

- 7 Find all permutations $(a_1, a_2, \dots, a_{2021})$ of $(1, 2, \dots, 2021)$, such that for every two positive integers m and n with difference bigger than 20^{21} , the following inequality holds: $GCD(m + 1, n + a_1) + GCD(m + 2, n + a_2) + \dots + GCD(m + 2021, n + a_{2021}) < 2|m - n|$.
-

- 8 Each girl among 100 girls has 100 balls; there are in total 10000 balls in 100 colors, from each color there are 100 balls. On a move, two girls can exchange a ball (the first gives the second one of her balls, and vice versa). The operations can be made in such a way, that in the end, each girl has 100 balls, colored in the 100 distinct colors. Prove that there is a sequence of operations, in which each ball is exchanged no more than 1 time, and at the end, each girl has 100 balls, colored in the 100 colors.
-