Finals 2021

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– Day 1

1 Let p_i for $i = 1, 2, ..., k$ be a sequence of smallest consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 3$ etc.). Let $N = p_1 \cdot p_2 \cdot ... \cdot p_k$. Prove that in a set $\{1, 2, ..., N\}$ there exist exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

For $k = 2$ $p_1 = 2$, $p_2 = 3$, $N = 6$. So in set $\{1, 2, 3, 4, 5, 6\}$ we can find 3 number satisfying thesis: 2, 3 and 4. (1 and 5 are not divisible by 2 or 3, and 6 is divisible by both of them so by even number of primes)

2 Let n be an integer. For pair of integers $0 \leq i, j \leq n$ there exist real number $f(i, j)$ such that:

1) $f(i, i) = 0$ for all integers $0 \leq i \leq n$

2) $0 \leq f(i, l) \leq 2\max\{f(i, j), f(j, k), f(k, l)\}$ for all integers i, j, k, l satisfying $0 \leq i \leq j \leq k \leq l \leq n$.

Prove that

$$f(0, n) \leq 2\sum_{k=1}^{n} f(k-1, k)$$

3 Let ω be the circumcircle of a triangle ABC. Let P be any point on ω different than the verticies of the triangle.
Line AP intersects BC at D, BP intersects AC at E and CP intersects AB at F. Let X be the projection of D onto line passing through midpoints of AP and BC, Y be the projection of E onto line passing through BP and AC and let Z be the projection of F onto line passing through midpoints of CP and AB. Let Q be the circumcenter of triangle XYZ. Prove that all possible points Q, corresponding to different positions of P lie on one circle.

– Day 2

4 Prove that for every pair of positive real numbers a, b and for every positive integer n,

$$(a+b)^n - a^n - b^n \geq \frac{2^n - 2}{2^{n-2}} \cdot ab(a+b)^{n-2}.$$

5 A convex hexagon $ABCDEF$ is given where $\measuredangle FAB + \measuredangle BCD + \measuredangle DEF = 360^\circ$ and $\measuredangle AEB = \measuredangle ADB$. Suppose the lines AB and DE are not parallel. Prove that the circumcenters of the triangles $\triangle AFE, \triangle BCD$ and the intersection of the lines AB and DE are collinear.

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6 Given an integer $d \ge 2$ and a circle ω. Hansel drew a finite number of chords of circle ω. The following condition is fulfilled: each end of each chord drawn is at least an end of d different drawn chords. Prove that there is a drawn chord which intersects at least $\frac{d^2}{4}$ other drawn chords. Here we assume that the chords with a common end intersect.

Note: Proof that a certain drawn chord crosses at least $\frac{d^2}{8}$ other drawn chords will be awarded two points.

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