Romania National Olympiad 2021

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–	Grade 7
1	Let $\mathcal{C}$ be a circle centered at O and $A \neq O$ be a point in its interior. The perpendicular bisector of the segment OA meets $\mathcal{C}$ at the points B and C, and the lines AB and AC meet $\mathcal{C}$ again at D and E, respectively. Show that the circles (OBC) and (ADE) have the same centre. *Ion Pătrașcu, Ion Cotoi*
2	Solve the system in reals: $\frac{4-a}{b} = \frac{5-b}{a} = \frac{10}{a^2+b^2}$.
3	Let ABC be a scalene triangle with $\angle BAC > 90^\circ$. Let D and E be two points on the side BC such that $\angle BAD = \angle ACB$ and $\angle CAE = \angle ABC$. The angle-bisector of $\angle ACB$ meets AD at N, If $MN \parallel BC$, determine $\angle(BM, CN)$. *Petru Braica*
4	Determine the smallest non-negative integer n such that $$\sqrt{(6n+11)(6n+14)(20n+19)} \in \mathbb{Q}.$$ *Mihai Bunget*
–	Grade 8
1	In the cuboid $ABCDA'B'C'D'$ with $AB = a$, $AD = b$ and $AA' = c$ such that $a > b > c > 0$, the points E and F are the orthogonal projections of A on the lines $A'D$ and $A'B$, respectively, and the points M and N are the orthogonal projections of C on the lines $C'D$ and $C'B$, respectively. Let $DF \cap BE = \{G\}$ and $DN \cap BM = \{P\}$. - Show that $(A'AG) \parallel (C'CP)$ and determine the distance between these two planes; - Show that $GP \parallel (ABC)$ and determine the distance between the line GP and the plane (ABC). *Petre Simion, Nicolae Victor Ioan*
2	Prove that for all positive real numbers a, b, c the following inequality holds: $$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{2(a^2+b^2+c^2)}{ab+bc+ca} + 7$$

and determine all cases of equality.

Lucian Petrescu

3 Solve the system in reals: $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=2022$ and $x+y=\frac{2021}{\sqrt{2022}}$

4 Students in a class of n students had to solve 2^{n-1} problems on an exam. It turned out that for each pair of distinct problems:
• there is at least one student who has solved both
• there is at least one student who has solved one of them, but not the other.
Show that there is a problem solved by all the students in the class.

– Grade 9

1 Let ABC be an acute-angled triangle with the circumcenter O. Let D be the foot of the altitude from A. If $OD \parallel AB$, show that $\sin 2B = \cot C$.

Mădălin Mitrofan

2 Let $P_0, P_1, \ldots, P_{2021}$ points on the unit circle of centre O such that for each $n \in \{1, 2, \ldots, 2021\}$ the length of the arc from P_{n-1} to P_n (in anti-clockwise direction) is in the interval $\left[\frac{\pi}{2}, \pi\right]$. Determine the maximum possible length of the vector:

$$\overrightarrow{OP_0} + \overrightarrow{OP_1} + \ldots + \overrightarrow{OP_{2021}}.$$

Mihai Iancu

3 If $a, b, c > 0, a+b+c=1$,then:

$\frac{1}{abc} + \frac{4}{a^2+b^2+c^2} \geq \frac{13}{ab+bc+ca}$

4 Let A be a finite set of non-negative integers. Determine all functions $f : \mathbb{Z}_{\geq 0} \to A$ such that

$$f(|x-y|) = |f(x) - f(y)|$$

for each $x, y \in \mathbb{Z}_{\geq 0}$.

Andrei Bâra

– Grade 10

1 Find the complex numbers x, y, z,with $\mid x \mid=\mid y \mid=\mid z \mid$,knowing that

$x+y+z$ and $x^3+y^3+z^3$ are be real numbers.

2 Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$, $d \neq 0$ and the function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$f(n) = \left\lfloor \frac{an+b}{cn+d} \right\rfloor \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Prove that the following are equivalent:

- f is surjective;
- $c = 0$, $b < d$ and $0 < a \leq d$.

Tiberiu Trif

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3 Let $n \geq 2$ be a positive integer such that the set of nth roots of unity has less than $2^{\lfloor \sqrt{n} \rfloor} - 1$ subsets with the sum 0. Show that n is a prime number.

Cristi Săvescu

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4 Determine all nonzero integers a for which there exists two functions $f, g : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + g(y)) = g(x) + f(y) + ay \text{ for all } x, y \in \mathbb{Q}.$$

Also, determine all pairs of functions with this property.

Vasile Pop

– Grade 11

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1 Let $f : [a, b] \to \mathbb{R}$ a function with Intermediate Value property such that $f(a) * f(b) < 0$. Show that there exist α, β such that $a < \alpha < \beta < b$ and $f(\alpha) + f(\beta) = f(\alpha) * f(\beta)$.

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2 Let $n \geq 2$ and $a_1, a_2, \ldots, a_n$, nonzero real numbers not necessarily distinct. We define matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$, $a_{i,j} = max\{a_i, a_j\}$, $\forall i, j \in \{1, 2, \ldots, n\}$. Show that $\mathbf{rank}(A)$= **card** $\{a_k | k = 1, 2, \ldots n\}$

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3 Let $f : \mathbb{R} \to \mathbb{R}$ a function $n \geq 2$ times differentiable so that: $\lim_{x\to\infty} f(x) = l \in \mathbb{R}$ and $\lim_{x\to\infty} f^{(n)}(x) = 0$.
Prove that: $\lim_{x\to\infty} f^{(k)}(x) = 0$ for all $k \in \{1, 2, \ldots, n-1\}$, where $f^{(k)}$ is the k - th derivative of f.

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4 Let $n \geq 2$ and matrices $A, B \in M_n(\mathbb{R})$. There exist $x \in \mathbb{R} \backslash \{0, \frac{1}{2}, 1\}$, such that $xAB + (1-x)BA = I_n$. Show that $(AB - BA)^n = O_n$.

– Grade 12

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1 Find all continuous functions $f:[0,1]\to[0,\infty)$ such that:

$\int_0^1 f\left(x\right)dx\cdot\int_0^1 f^2\left(x\right)dx\cdot\ldots\cdot\int_0^1 f^{2020}\left(x\right)dx=\left(\int_0^1 f^{2021}\left(x\right)dx\right)^{1010}$

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2 Determine all non-trivial finite rings with am unit element in which the sum of all elements is invertible.

Mihai Opincariu

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3 Given is an positive integer $a>2$
a) Prove that there exists positive integer n different from 1, which is not a prime, such that $a^n=1(mod n)$
b) Prove that if p is the smallest positive integer, different from 1, such that $a^p=1(mod p)$, then p is a prime.
c) There does not exist positive integer n, different from 1, such that $2^n=1(mod n)$

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4 Let be $f:[0,1]\to[0,1]$ a continuous and bijective function,such that :

$f\left(0\right)=0$.Then the following inequality holds:

$\left(\alpha+2\right)\cdot\int_0^1 x^{\alpha}\left(f\left(x\right)+f^{-1}\left(x\right)\right)\le 2,\forall\alpha\ge 0$
