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– Test 1

Problem 1 Find all functions f defined on real numbers and taking values in the set of real numbers such that $f(x+y)+f(y+z)+f(z+x)\geq f(x+2y+3z)$ for all real numbers x, y, z.

There is an infinity of such functions. Every function with the property that $3\inf f \geq \sup f$ is a good one. I wonder if there is a way to find all the solutions. It seems very strange.

Problem 2 Let $f(n)$ denote the least positive integer k such that $1+2+\cdots+k$ is divisible by n. Show that $f(n)=2n-1$ if and only if n is a power of 2.

Problem 3 For which positive integers n is there a permutation $(x_1, x_2, \ldots, x_n)$ of $1, 2, \ldots, n$ such that all the differences $|x_k - k|$, $k = 1, 2, \ldots, n$, are distinct?

Problem 4 Let ABC be a triangle with circumcenter O. Let P and Q be points on the segments AB and AC, respectively, such that $BP : PQ : QC = AC : CB : BA$.
Prove that the points A, P, Q and O lie on one circle.

Alternative formulation. Let O be the center of the circumcircle of a triangle ABC. If P and Q are points on the sides AB and AC, respectively, satisfying $\frac{BP}{PQ}=\frac{CA}{BC}$ and $\frac{CQ}{PQ}=\frac{AB}{BC}$, then show that the points A, P, Q and O lie on one circle.

– Test 2

Problem 1 given that p,q are two polynomials such that each one has at least one root and

$$p(1+x+q(x)^2)=q(1+x+p(x)^2)$$

then prove that p=q

Problem 2 A set S consists of k sequences of $0,1,2$ of length n. For any two sequences $(a_i),(b_i)\in S$ we can construct a new sequence (c_i) such that $c_i=\left\lfloor\frac{a_i+b_i+1}{2}\right\rfloor$ and include it in S. Assume that after performing finitely many such operations we obtain all the $3n$ sequences of $0,1,2$ of length n. Find the least possible value of k.

Problem 3 In a triangle ABC, the internal and external bisectors of the angle A intersect the line BC at D and E respectively. The line AC meets the circle with diameter DE again at F. The tangent line to the circle ABF at A meets the circle with diameter DE again at G. Show that $AF = AG$.

Problem 4 Prove that for all integers $n \geq 3$ there exists a set $A_n = \{a_1, a_2, \ldots, a_n\}$ of n distinct natural numbers such that, for each $i = 1, 2, \ldots, n$,

$$\prod_{\substack{1 \leq k \leq n \\ k \neq i}} a_k \equiv 1 \pmod{a_i}.$$
