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– Test 1

Problem 1 Find all positive integers n with the following property: There exists a positive integer k and mutually distinct integers $x_1, x_2, \ldots, x_n$ such that the set $\{x_i + x_j \mid 1 \le i < j \le n\}$ is a set of distinct powers of k.

Problem 2 If a, b, c, d are Distinct Real no. such that

$$a = \sqrt{4 + \sqrt{5 + a}}$$
$$b = \sqrt{4 - \sqrt{5 + b}}$$
$$c = \sqrt{4 + \sqrt{5 - c}}$$
$$d = \sqrt{4 - \sqrt{5 - d}}$$

Then $abcd =$

Problem 3 Let BD and CE be the bisectors of the interior angles $\angle B$ and $\angle C$, respectively ($D \in AC$, $E \in AB$). Consider the circumcircle of ABC with center O and the excircle corresponding to the side BC with center I_a. These two circles intersect at points P and Q.

(a) Prove that PQ is parallel to DE.
(b) Prove that I_aO is perpendicular to DE.

Problem 4 Let Q+ and Z denote the set of positive rationals and the set of integers, respectively. Find all functions f : Q+ Z satisfying the following conditions:
(i) f(1999) = 1;
(ii) f(ab) = f(a) + f(b) for all a, b Q+;
(iii) f(a + b) minf(a), f(b) for all a, b Q+.

Problem 5 (a) If m, n are positive integers such that $2^n - 1$ divides $m^2 + 9$, prove that n is a power of 2;
(b) If n is a power of 2, prove that there exists a positive integer m such that $2^n - 1$ divides $m^2 + 9$.

– Test 2

Problem 1 For a positive integer n, let $w(n)$ denote the number of distinct prime divisors of n. Determine the least positive integer k such that $2^{w(n)} \leq k\sqrt[4]{n}$ for all positive integers n.

Problem 2 In a triangle ABC, the bisector of the angle at A of a triangle ABC intersects the segment BC and the circumcircle of ABC at points A_1 and A_2, respectively. Points B_1, B_2, C_1, C_2 are analogously defined. Prove that

$$\frac{A_1A_2}{BA_2 + CA_2} + \frac{B_1B_2}{CB_2 + AB_2} + \frac{C_1C_2}{AC_2 + BC_2} \geq \frac{3}{4}.$$

Problem 3 A sequence a_n is defined by

$$a_0 = 0, \qquad a_1 = 3;$$

$$a_n = 8a_{n-1} + 9a_{n-2} + 16 \text{ for } n \geq 2.$$

Find the least positive integer h such that $a_{n+h} - a_n$ is divisible by 1999 for all $n \geq 0$.

Problem 4 Assume that it is possible to color more than half of the surfaces of a given polyhedron so that no two colored surfaces have a common edge.

(a) Describe one polyhedron with the above property.
(b) Prove that one cannot inscribe a sphere touching all the surfaces of a polyhedron with the above property.
