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by parmenides51

1 For $r > 0$ denote by B_r the set of points at distance at most r length units from the origin. If P_r is the set of the points in B_r whit integer coordinates, show that the equation

$$xy^3z + 2x^3z^3 - 3x^5y = 0$$

has an odd number of solutions (x, y, z) in P_r.

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2 The paper folding art origami is usually performed with square sheets of paper. Someone folds the sheet once along a line through the center of the sheet in orde to get a nonagon. Let p be the perimeter of the nonagon minus the length of the fold, i.e. the total length of the eight sides that are not folds, and denote by s the original side length of the square. Express the area of the nonagon in terms of p and s.

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3 Determine all primes p and all non-negative integers m and n, such that

$$1 + p^n = m^3.$$

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4 A robotic lawnmower is located in the middle of a large lawn. Due a manufacturing defect, the robot can only move straight ahead and turn in directions that are multiples of 60^o. A fence must be set up so that it delimits the entire part of the lawn that the robot can get to, by traveling along a curve with length no more than 10 meters from its starting position, given that it is facing north when it starts. How long must the fence be?

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5 Let $n \geq 2$ be a positive integer. Show that there are exactly $2^{n-3}n(n-1)$ n-tuples of integers $(a_1, a_2, \ldots, a_n)$, which satisfy the conditions:
(i) $a_1 = 0$;
(ii) for each m, $2 \leq m \leq n$, there is an index in m, $1 \leq i_m < m$, such that $|a_{i_m} - a_m| \leq 1$;
(iii) the n-tuple $(a_1, a_2, \ldots, a_n)$ contains exactly $n-1$ different numbers.

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6 Let a, b, c, be real numbers such that

$$a^2b^2 + 18abc > 4b^3 + 4a^3c + 27c^2.$$

Prove that $a^2 > 3b$.

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