HMMT Invitational Competition 2021

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by numbersandnumbers

1 2021 people are sitting around a circular table. In one move, you may swap the positions of two people sitting next to each other. Determine the minimum number of moves necessary to make each person end up 1000 positions to the left of their original position.

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2 Let n be a positive integer. Alice writes n real numbers $a_1, a_2, \dots, a_n$ in a line (in that order). Every move, she picks one number and replaces it with the average of itself and its neighbors (a_n is not a neighbor of a_1, nor vice versa). A number *changes sign* if it changes from being nonnegative to negative or vice versa. In terms of n, determine the maximum number of times that a_1 can change sign, across all possible values of $a_1, a_2, \dots, a_n$ and all possible sequences of moves Alice may make.

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3 Let A be a set of $n \geq 2$ positive integers, and let $f(x) = \sum_{a \in A} x^a$. Prove that there exists a complex number z with $|z| = 1$ and $|f(z)| = \sqrt{n-2}$.

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4 Let $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, and $C_1C_2C_3C_4$ be three regular tetrahedra in 3-dimensional space, no two of which are congruent. Suppose that, for each $i \in \{1, 2, 3, 4\}$, C_i is the midpoint of the line segment A_iB_i. Determine whether the four lines A_1B_1, A_2B_2, A_3B_3, and A_4B_4 must concur.

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5 In an $n \times n$ square grid, n squares are marked so that every rectangle composed of exactly n grid squares contains at least one marked square. Determine all possible values of n.
