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by jasperE3, Babai, parmenides51

– 1st Grade

Problem 1 Find all positive integer solutions x, y, z such that $1/x + 2/y - 3/z = 1$

Problem 2 The incircle of a triangle ABC touches BC, CA, AB at A_1, B_1, C_1, respectively. Find the angles of $\triangle A_1B_1C_1$ in terms of the angles of $\triangle ABC$.

Problem 3 Let $m > 1$ be an integer. Determine the number of positive integer solutions of the equation $\left\lfloor \frac{x}{m} \right\rfloor = \left\lfloor \frac{x}{m-1} \right\rfloor$.

Problem 4 We are given coins of $1, 2, 5, 10, 20, 50$ lipas and of 1 kuna (Croatian currency: 1 kuna = 100 lipas). Prove that if a bill of M lipas can be paid by N coins, then a bill of N kunas can be paid by M coins.

– 2nd Grade

Problem 1 Let $a > 0$ and x_1, x_2, x_3 be real numbers with $x_1 + x_2 + x_3 = 0$. Prove that

$$\log_2 (1 + a^{x_1}) + \log_2 (1 + a^{x_2}) + \log_2 (1 + a^{x_3}) \geq 3.$$

Problem 2 Two squares $ACXE$ and $CBDY$ are constructed in the exterior of an acute-angled triangle ABC. Prove that the intersection of the lines AD and BE lies on the altitude of the triangle from C.

Problem 3 Let j and k be integers. Prove that the inequality

$$\lfloor (j+k)\alpha \rfloor + \lfloor (j+k)\beta \rfloor \geq \lfloor j\alpha \rfloor + \lfloor j\beta \rfloor + \lfloor k(\alpha+\beta) \rfloor$$

holds for all real numbers α, β if and only if $j = k$.

Problem 4 Let $ABCD$ be a square with side 20 and $T_1, T_2, ..., T_{2000}$ are points in $ABCD$ such that no 3 points in the set $S = \{A, B, C, D, T_1, T_2, ..., T_{2000}\}$ are collinear. Prove that there exists a triangle with vertices in S, such that the area is less than $1/10$.

– 3rd Grade

Problem 1 Let B and C be fixed points, and let A be a variable point such that $\angle BAC$ is fixed. The midpoints of AB and AC are D and E respectively, and F, G are points such that $DF \perp AB$, $EG \perp AC$ and BF and CG are perpendicular to BC. Prove that $BF \cdot CG$ remains constant as A varies.

Problem 2 Find all 5-tuples of different four-digit integers with the same initial digit such that the sum of the five numbers is divisible by four of them.

Problem 3 A plane intersects a rectangular parallelepiped in a regular hexagon. Prove that the rectangular parallelepiped is a cube.

Problem 4 If $n \geq 2$ is an integer, prove the equality

$$\lfloor \log_2 n \rfloor + \lfloor \log_3 n \rfloor + \ldots + \lfloor \log_n n \rfloor = \left\lfloor \sqrt{n} \right\rfloor + \left\lfloor \sqrt[3]{n} \right\rfloor + \ldots + \left\lfloor \sqrt[n]{n} \right\rfloor .$$

– 4th Grade

Problem 1 Let $\mathcal{P}$ be the parabola $y^2 = 2px$, and let T_0 be a point on it. Point T_0' is such that the midpoint of the segment T_0T_0' lies on the axis of the parabola. For a variable point T on $\mathcal{P}$, the perpendicular from T_0' to the line T_0T intersects the line through T parallel to the axis of $\mathcal{P}$ at a point T'. Find the locus of T'.

Problem 2 Let ABC be a triangle with $AB = AC$. With center in a point of the side BC, the circle S is constructed that is tangent to the sides AB and AC. Let P and Q be any points on the sides AB and AC respectively, such that PQ is tangent to S. Show that $PB \cdot CQ = \left(\frac{BC}{2}\right)^2$

Problem 3 Let $n \geq 3$ positive integers $a_1, \ldots, a_n$ be written on a circle so that each of them divides the sum of its two neighbors. Let us denote

$$S_n = \frac{a_n + a_2}{a_1} + \frac{a_1 + a_3}{a_2} + \ldots + \frac{a_{n-2} + a_n}{a_{n-1}} + \ldots + \frac{a_{n-1} + a_1}{a_n}.$$

Determine the minimum and maximum values of S_n.

Problem 4 Let S be the set of all squarefree numbers and n be a natural number. Prove that

$$\sum_{k \in S} \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor = n.$$