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– Test 1

Problem 1 Let N be a positive integer greater than 2. We number the vertices of a regular 2n-gon clockwise with the numbers 1, 2, . . . ,N,N,N + 1, . . . ,2,1. Then we proceed to mark the vertices in the following way. In the first step we mark the vertex 1. If ni is the vertex marked in the i-th step, in the i+1-th step we mark the vertex that is —ni— vertices away from vertex ni, counting clockwise if ni is positive and counter-clockwise if ni is negative. This procedure is repeated till we reach a vertex that has already been marked. Let $f(N)$ be the number of non-marked vertices.
(a) If $f(N) = 0$, prove that 2N + 1 is a prime number.
(b) Compute $f(1997)$.

Problem 2 Suppose that S is a finite set of real numbers with the property that any two distinct elements of S form an arithmetic progression with another element in S. Give an example of such a set with 5 elements and show that no such set exists with more than 5 elements.

Problem 3 Let $\mathbb{N}$ be the set of positive integers. Find all functions defined on $\mathbb{N}$ and taking values on $\mathbb{N}$ satisfying, for all $n \in \mathbb{N}$,

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1998.$$

Problem 4 Let L be a circle with center O and tangent to sides AB and AC of a triangle ABC in points E and F, respectively. Let the perpendicular from O to BC meet EF at D. Prove that A, D and M are collinear, where M is the midpoint of BC.

Problem 5 Consider k positive integers $a_1, a_2, \dots, a_k$ satisfying $1 \le a_1 < a_2 < \dots < a_k \le n$ and $\operatorname{lcm}(a_i, a_j) \le n$ for any i, j. Prove that

$$k \le 2\lfloor \sqrt{n} \rfloor.$$

– Test 2

Problem 1 Let ABC be an acute-angled triangle. Construct three semi-circles, each having a different side of ABC as diameter, and outside ABC. The perpendiculars dropped from A, B, C to the opposite sides intersect these semi-circles in points E, F, G, respectively. Prove that the hexagon $AGBECF$ can be folded so as to form a pyramid having ABC as base.

Problem 2 There are $n \geq 3$ integers around a circle. We know that for each of these numbers the ratio between the sum of its two neighbors and the number is a positive integer. Prove that the sum of the n ratios is not greater than $3n$.

Problem 3 Show that it is possible to color the points of $\mathbb{Q} \times \mathbb{Q}$ in two colors in such a way that any two points having distance 1 have distinct colors.

Problem 4 (a) Show that, for each positive integer n, the number of monic polynomials of degree n with integer coefficients having all its roots on the unit circle is finite.
(b) Let $P(x)$ be a monic polynomial with integer coefficients having all its roots on the unit circle. Show that there exists a positive integer m such that $y^m = 1$ for each root y of $P(x)$.

Problem 5 Let p be an odd prime integer and k a positive integer not divisible by p, $1 \leq k < 2(p+1)$, and let $N = 2kp + 1$. Prove that the following statements are equivalent:

(i) N is a prime number
(ii) there exists a positive integer a, $2 \leq a < n$, such that $a^{kp} + 1$ is divisible by N and $\gcd\left(a^k + 1, N\right) = 1$.