**Final Round - 2021 Korea**

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**Part** 1

**P1** An acute triangle $\triangle ABC$ and its incenter $I$, circumcenter $O$ is given. The line that is perpendicular to $AI$ and passes $I$ intersects with $AB$, $AC$ in $D$,$E$. The line that is parallel to $BI$ and passes $D$ and the line that is parallel to $CI$ and passes $E$ intersects in $F$. Denote the circumcircle of $DEF$ as $\omega$, and its center as $K$. $\omega$ and $FI$ intersect in $P(\neq F)$. Prove that $O, K, P$ is collinear.

**P2** Positive integer $k(\geq 8)$ is given. Prove that if there exists a pair of positive integers $(x, y)$ that satisfies the conditions below, then there exists infinitely many pairs $(x, y)$.
(1) $x \mid y^2 - 3, y \mid x^2 - 2$
(2) $gcd\left(3x + \frac{2(y^2-3)}{x}, 2y + \frac{3(x^2-2)}{y}\right) = k$

**P3** Let $P$ be a set of people. For two people $A$ and $B$, if $A$ knows $B$, $B$ also knows $A$. Each person in $P$ knows $2$ or less people in the set. $S$, a subset of $P$ with $k$ people, is called ***k-independent set*** of $P$ if any two people in $S$ don't know each other. $X_1, X_2, \dots, X_{4041}$ are ***2021-independent set***s of $P$ (not necessarily distinct). Show that there exists a ***2021-independent set*** of $P$, $\{v_1, v_2, \dots, v_{2021}\}$, which satisfies the following condition:

For some integer $1 \leq i_1 < i_2 < \cdots < i_{2021} \leq 4041$, $v_1 \in X_{i_1}, v_2 \in X_{i_2}, \dots, v_{2021} \in X_{i_{2021}}$

Thanks to Evan Chen, here's a graph wording of the problem :)

Let $G$ be a finite simple graph with maximum degree at most $2$. Let $X_1, X_2, \dots, X_{4041}$ be independent sets of size $2021$ *(not necessarily distinct)*. Prove that there exists another independent set $\{v_1, v_2, \dots, v_{2021}\}$ of size $2021$ and indices $1 \leq t_1 < t_2 < \cdots < t_{2021} \leq 4041$ such that $v_i \in X_{t_i}$ for all $i$.

**Part** 2

**P4** There are $n(\geq 2)$ clubs $A_1, A_2, ...A_n$ in Korean Mathematical Society. Prove that there exist $n-1$ sets $B_1, B_2, ...B_{n-1}$ that satisfy the condition below.
(1) $A_1 \cup A_2 \cup \cdots A_n = B_1 \cup B_2 \cup \cdots B_{n-1}$
(2) for any $1 \leq i < j \leq n-1$, $B_i \cap B_j = \emptyset, -1 \leq |B_i| - |B_j| \leq 1$
(3) for any $1 \leq i \leq n-1$, there exist $A_k, A_j (1 \leq k \leq j \leq n)$such that $B_i \subseteq A_k \cup A_j$

**P5** The incenter and $A$-excenter of $\triangle ABC$ is $I$ and $O$. The foot from $A, I$ to $BC$ is $D$ and $E$. The intersection of $AD$ and $EO$ is $X$. The circumcenter of $\triangle BXC$ is $P$.
Show that the circumcircle of $\triangle BPC$ is tangent to the $A$-excircle if $X$ is on the incircle of $\triangle ABC$.

**P6** Find all functions $f, g : \mathbb{R} \to \mathbb{R}$ such that satisfies

$$f(x^2 - g(y)) = g(x)^2 - y$$

for all $x, y \in \mathbb{R}$

**Timeline** 3h/3h