Spain Mathematical Olympiad 2021

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–	Day 1
1	Vertices A, B, C of a equilateral triangle of side 1 are in the surface of a sphere with radius 1 and center O. Let D be the orthogonal projection of A on the plane α determined by points B, C, O. Let N be one of the intersections of the line perpendicular to α passing through O with the sphere. Find the angle $\angle DNO$.
2	Given a positive integer n, we define $\lambda(n)$ as the number of positive integer solutions of $x^2 - y^2 = n$. We say that n is *olympic* if $\lambda(n) = 2021$. Which is the smallest olympic positive integer? Which is the smallest olympic positive odd integer?
3	We have 2021 colors and 2021 chips of each color. We place the 2021^2 chips in a row. We say that a chip F is *bad* if there is an odd number of chips that have a different color to F both to the left and to the right of F. (a) Determine the minimum possible number of bad chips. (b) If we impose the additional condition that each chip must have at least one adjacent chip of the same color, determine the minimum possible number of bad chips.
–	Day 2
4	Let a, b, c, d real numbers such that: $$a + b + c + d = 0 \text{ and } a^2 + b^2 + c^2 + d^2 = 12$$ Find the minimum and maximum possible values for $abcd$, and determine for which values of a, b, c, d the minimum and maximum are attained.
5	We have $2n$ lights in two rows, numbered from 1 to n in each row. Some (or none) of the lights are on and the others are off, we call that a "state". Two states are distinct if there is a light which is on in one of them and off in the other. We say that a state is good if there is the same number of lights turned on in the first row and in the second row. Prove that the total number of good states divided by the total number of states is: $$\frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n n!}$$

6 Let ABC be a triangle with $AB \neq AC$, let I be its incenter, γ its inscribed circle and D the midpoint of BC. The tangent to γ from D different to BC touches γ in E. Prove that AE and DI are parallel.