

ITAMO 2021

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- 1 A positive integer m is said to be *zero taker* if there exists a positive integer k such that:
 - k is a perfect square; m divides k ;
 - the decimal expression of k contains at least 2021 '0' digits, but the last digit of k is not equal to 0.
 Find all positive integers that are zero takers.

- 2 Let ABC a triangle and let I be the center of its inscribed circle. Let D be the symmetric point of I with respect to AB and E be the symmetric point of I with respect to AC . Show that the circumcircles of the triangles BID and CIE are each other tangent.

- 3 A grid consists of $n \times n$ points, with $n \in \mathbb{Z}^+$. In some of these points is a sentry. Every sentry chooses two directions, one perpendicular to the other (one vertical and the other horizontal) and watches over all the points that are found in the two chosen directions. Each sentry watches over her own point as well and the sentries on the edge of the grid can also watch the vacuum, depending on the directions they have chosen.

For instance, in the figure below, representing a disposition of 5 sentries in a 4×4 grid, the sentries in A, B, C, D, E watch over 1, 3, 4, 5, 7 points, respectively; points D and E are watched by one sentry, point C is watched by two sentries, points A, B and F are watched by three sentries.

(a) Prove that we can place 12 sentries in a 4×4 grid in such a way that every point of the grid is watched by at most two sentries.

(b) Let $S(n)$ be the maximum number of sentries we can place on an $n \times n$ grid in such a way that every point of the grid is watched by at most two sentries. Prove that $3n \leq S(n) \leq 4n$ for all $n \geq 3$.

- 4 Given two fractions a/b and c/d we define their *pirate sum* as: $\frac{a}{b} \star \frac{c}{d} = \frac{a+c}{b+d}$ where the two initial fractions are simplified the most possible, like the result.

For example, the pirate sum of $2/7$ and $4/5$ is $1/2$.

Given an integer $n \geq 3$, initially on a blackboard there are the fractions: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$.

At each step we choose two fractions written on the blackboard, we delete them and write at their place their pirate sum. Continue doing the same thing until on the blackboard there is only one fraction.

Determine, in function of n , the maximum and the minimum possible value for the last fraction.

- 5 Let ABC be an acute-angled triangle, let M be the midpoint of BC and let H be the foot of the B -altitude. Let Q be the circumcenter of ABM and let X be the intersection point between BH and the axis of BC .

Show that the circumcircles of the two triangles ACM , AXH and the line CQ pass through a same point if and only if BQ is perpendicular to CQ .

- 6 A sequence $x_1, x_2, \dots, x_n, \dots$ consists of an initial block of p positive distinct integers that then repeat periodically. This means that $\{x_1, x_2, \dots, x_p\}$ are p distinct positive integers and $x_{n+p} = x_n$ for every positive integer n . The terms of the sequence are not known and the goal is to find the period p . To do this, at each move it possible to reveal the value of a term of the sequence at your choice.

(a) Knowing that $1 \leq p \leq 10$, find the least n such that there is a strategy which allows to find p revealing at most n terms of the sequence.

(b) Knowing that p is one of the first k prime numbers, find for which values of k there exist a strategy that allows to find p revealing at most 5 terms of the sequence.
