Serbia National Math Olympiad 2021

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– Day 1

1 Let $a > 1$ and c be natural numbers and let $b \neq 0$ be an integer. Prove that there exists a natural number n such that the number $a^n + b$ has a divisor of the form $cx + 1$, $x \in \mathbb{N}$.

2 In the country of Graphia there are 100 towns, each numbered from 1 to 100. Some pairs of towns may be connected by a (direct) road and we call such pairs of towns *adjacent*. No two roads connect the same pair of towns.

Peter, a foreign tourist, plans to visit Graphia 100 times. For each i, $i = 1, 2, \ldots, 100$, Peter starts his i-th trip by arriving in the town numbered i and then each following day Peter travels from the town he is currently in to an adjacent town with the lowest assigned number, assuming such that a town exists and that he hasn't visited it already on the i-th trip. Otherwise, Peter deems his i-th trip to be complete and returns home.

It turns out that after all 100 trips, Peter has visited each town in Graphia the same number of times. Find the largest possible number of roads in Graphia.

3 In a triangle ABC, let AB be the shortest side. Points X and Y are given on the circumcircle of $\triangle ABC$ such that $CX = AX + BX$ and $CY = AY + BY$. Prove that $\measuredangle XCY < 60^o$.

– Day 2

4 A convex quadrilateral $ABCD$ will be called *rude* if there exists a convex quadrilateral $PQRS$ whose points are all in the interior or on the sides of quadrilateral $ABCD$ such that the sum of diagonals of $PQRS$ is larger than the sum of diagonals of $ABCD$.

Let $r > 0$ be a real number. Let us assume that a convex quadrilateral $ABCD$ is not rude, but every quadrilateral $A'BCD$ such that $A' \neq A$ and $A'A \leq r$ is rude. Find all possible values of the largest angle of $ABCD$.

5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$ the following equality holds:

$$f(xf(y) + x^2 + y) = f(x)f(y) + xf(x) + f(y).$$

6 A finite sequence of natural numbers $a_1, a_2, \ldots, a_n$ is given. A sub-sequence $a_{k+1}, a_{k+2}, \ldots, a_l$ will be called a *repetition* if there exists a natural number $p \leq \frac{l-k}{2}$ such that $a_i = a_{i+p}$ for $k + 1 \leq i \leq l - p$, but $a_i \neq a_{i+p}$ for $i = k$ (if $k > 0$) and $i = l - p + 1$ (if $l < n$).

Show that the sequence contains less than n repetitions.