Serbia National Math Olympiad 2020

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–	Day 1 (August 24, 2020)
1	Find all monic polynomials $P(x)$ such that the polynomial $P(x)^2 - 1$ is divisible by the polynomial $P(x+1)$.
2	We are given a polyhedron with at least 5 vertices, such that exactly 3 edges meet in each of the vertices. Prove that we can assign a rational number to every vertex of the given polyhedron such that the following conditions are met: (i) At least one of the numbers assigned to the vertices is equal to 2020. (ii) For every polygonal face, the product of the numbers assigned to the vertices of that face is equal to 1.
3	We are given a triangle ABC. Points D and E on the line AB are such that $AD = AC$ and $BE = BC$, with the arrangment of points $D - A - B - E$. The circumscribed circles of the triangles DBC and EAC meet again at the point $X \neq C$, and the circumscribed circles of the triangles DEC and ABC meet again at the point $Y \neq C$. Find the measure of $\angle ACB$ given the condition $DY + EY = 2XY$.
–	Day 2 (August 25, 2020)
4	In a trapezoid $ABCD$ such that the internal angles are not equal to 90°, the diagonals AC and BD intersect at the point E. Let P and Q be the feet of the altitudes from A and B to the sides BC and AD respectively. Circumscribed circles of the triangles CEQ and DEP intersect at the point $F \neq E$. Prove that the lines AP, BQ and EF are either parallel to each other, or they meet at exactly one point.
5	For a natural number n, with $v_2(n)$ we denote the largest integer $k \geq 0$ such that $2^k \| n$. Let us assume that the function $f\colon \mathbb{N} \to \mathbb{N}$ meets the conditions: (i) $f(x) \leq 3x$ for all natural numbers $x \in \mathbb{N}$. (ii) $v_2(f(x)+f(y)) = v_2(x+y)$ for all natural numbers $x, y \in \mathbb{N}$. Prove that for every natural number a there exists exactly one natural number x such that $f(x) = 3a$.
6	We are given a natural number k. Let us consider the following game on an infinite onedimensional board. At the start of the game, we distrubute n coins on the fields of the given board

(one field can have multiple coins on itself). After that, we have two choices for the following moves:

(i) We choose two nonempty fields next to each other, and we transfer all the coins from one of the fields to the other. (ii) We choose a field with at least 2 coins on it, and we transfer one coin from the chosen field to the $k-\text{th}$ field on the left , and one coin from the chosen field to the $k-\text{th}$ field on the right.

$(\mathbf{a})$ If $n \leq k+1$, prove that we can play only finitely many moves. $(\mathbf{b})$ For which values of k we can choose a natural number n and distribute n coins on the given board such that we can play infinitely many moves.