Serbia National Math Olympiad 2019

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– Day 1

1 Find all positive integers $n, n > 1$ for wich holds :
If $a_1, a_2, \ldots, a_k$ are all numbers less than n and relatively prime to n , and holds $a_1 < a_2 < \cdots < a_k$, then none of sums $a_i + a_{i+1}$ for $i = 1, 2, 3, \ldots k-1$ are divisible by 3.

2 For the sequence of real numbers $a_1, a_2, \ldots, a_k$ we say it is *invested* on the interval $[b, c]$ if there exists numbers $x_0, x_1, \ldots, x_k$ in the interval $[b, c]$ such that $|x_i - x_{i-1}| = a_i$ for $i = 1, 2, 3, \ldots k$. A sequence is *normed* if all its members are not greater than 1 . For a given natural n , prove :

a)Every *normed* sequence of length $2n+1$ is *invested* in the interval $\left[0, 2 - \frac{1}{2^n}\right]$.
b) there exists *normed* sequence of length $4n+3$ wich is not *invested* on $\left[0, 2 - \frac{1}{2^n}\right]$.

3 Let k be the circle inscribed in convex quadrilateral $ABCD$. Lines AD and BC meet at P ,and circumcircles of $\triangle PAB$ and $\triangle PCD$ meet in X . Prove that tangents from X to k form equal angles with lines AX and CX .

– Day 2

4 For a $\triangle ABC$, let A_1 be the symmetric point of the intersection of angle bisector of $\angle BAC$ and BC , where center of the symmetry is the midpoint of side BC, In the same way we define B_1 (on AC) and C_1 (on AB). Intersection of circumcircle of $\triangle A_1B_1C_1$ and line AB is the set $\{Z, C_1\}$, with BC is the set $\{X, A_1\}$ and with CA is the set $\{Y, B_1\}$. If the perpendicular lines from X, Y, Z on BC, CA and AB , respectively are concurrent , prove that $\triangle ABC$ is isosceles.

5 In the spherical shaped planet X there are $2n$ gas stations. Every station is paired with one other station ,
and every two paired stations are diametrically opposite points on the planet.
Each station has a given amount of gas. It is known that : if a car with empty (large enough) tank starting
from any station it is always to reach the paired station with the initial station (it can get extra gas during the journey).
Find all naturals n such that for any placement of $2n$ stations for wich holds the above condo-tions, holds:
there always a gas station wich the car can start with empty tank and go to all other stations on the planet.(Consider that the car consumes a constant amount of gas per unit length.)

6 Sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are defined with recurrent relations :

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+1} = \frac{2018}{n} a_n + a_{n-1} \quad \text{for} \quad n \geq 1$$

and

$$b_0 = 0, \quad b_1 = 1, \quad b_{n+1} = \frac{2020}{n} b_n + b_{n-1} \quad \text{for} \quad n \geq 1$$

Prove that :

$$\frac{a_{1010}}{1010} = \frac{b_{1009}}{1009}$$