National Math Olympiad (Second Round) 2021

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– Day 1

1 There are two distinct Points A and B on a line. We color a point P on segment AB, distinct from A, B and midpoint of segment AB to red. In each move , we can reflect one of the red point wrt A or B and color the midpoint of the resulting point and the point we reflected from (which is one of A or B) to red. For example , if we choose P and the reflection of P wrt to A is P' , then midpoint of AP' would be red. Is it possible to make the midpoint of AB red after a finite number of moves?

2 Call a positive integer n "Fantastic" if none of its digits are zero and it is possible to remove one of its digits and reach to an integer which is a divisor of n . (for example , 25 is fantastic , as if we remove digit 2 , resulting number would be 5 which is divisor of 25) Prove that the number of Fantastic numbers is finite.

3 Circle ω is inscribed in quadrilateral $ABCD$ and is tangent to segments BC, AD at E, F , respectively.DE intersects ω for the second time at X. if the circumcircle of triangle DFX is tangent to lines AB and CD , prove that quadrilateral $AFXC$ is cyclic.

– Day 2

4 n points are given on a circle ω. There is a circle with radius smaller than ω such that all these points lie inside or on the boundary of this circle. Prove that we can draw a diameter of ω with endpoints not belonging to the given points such that all the n given points remain in one side of the diameter.

5 1400 real numbers are given. Prove that one can choose three of them like x, y, z such that :

$$\left|\frac{(x-y)(y-z)(z-x)}{x^4+y^4+z^4+1}\right| < 0.009$$

6 Is it possible to arrange 1400 positive integer (not necessarily distinct) ,at least one of them being 2021 , around a circle such that any number on this circle equals to the sum of gcd of the two previous numbers and two next numbers? for example , if a, b, c, d, e are five consecutive numbers on this circle , $c = \gcd(a, b) + \gcd(d, e)$