EGMO Team Selection Test for Brazil 2021

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by mathisreal

1 Let $x_0, x_1, x_2, \dots$ be a infinite sequence of real numbers, such that the following three equalities are true:
I- $x_{2k} = (4x_{2k-1} - x_{2k-2})^2$, for $k \geq 1$
II- $x_{2k+1} = |\frac{x_{2k}}{4} - k^2|$, for $k \geq 0$
III- $x_0 = 1$
a) Determine the value of x_{2022}
b) Prove that there are infinite many positive integers k, such that $2021|x_{2k+1}$

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2 Let a, b, k be positive integers such that $gcd(a,b)^2 + lcm(a,b)^2 + a^2b^2 = 2020^k$
Prove that k is an even number.

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3 Let ABC be an acute-angled triangle with $AC > AB$, and Ω is your circumcircle. Let P be the midpoint of the arc BC of Ω (not containing A) and Q be the midpoint of the arc BC of Ω(containing the point A). Let M be the foot of perpendicular of Q on the line AC. Prove that the circumcircle of $\triangle AMB$ cut the segment AP in your midpoint.

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4 The ***duchess*** is a chess piece such that the duchess attacks all the cells in two of the four diagonals which she is contained(the directions of the attack can vary to two different duchesses). Determine the greatest integer n, such that we can put n duchesses in a table 8×8 and none duchess attacks other duchess.
Note: The attack diagonals can be "outside" the table; for instance, a duchess on the top-leftmost cell we can choose attack or not the main diagonal of the table 8×8.

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5 Let S be a set, such that for every positive integer n, we have $|S \cap T| = 1$, where $T = \{n, 2n, 3n\}$. Prove that if $2 \in S$, then $13824 \notin S$.

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6 A plane geometric figure of n sides with the vertices $A_1, A_2, A_3, \dots, A_n$ (A_i is adjacent to A_{i+1} for every i integer where $1 \leq i \leq n-1$ and A_n is adjacent to A_1) is called *brazilian* if:
I - The segment A_jA_{j+1} is equal to $(\sqrt{2})^{j-1}$, for every j with $1 \leq j \leq n-1$.
II- The angles $\angle A_kA_{k+1}A_{k+2} = 135^\circ$, for every k with $1 \leq k \leq n-2$.
Note 1: The figure can be convex or not convex, and your sides can be crossed.
Note 2: The angles are in counterclockwise.
a) Find the length of the segment A_nA_1 for a brazilian figure with $n = 5$.
b) Find the length of the segment A_nA_1 for a brazilian figure with $n \equiv 1 \pmod 4$.

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7 The incircle ω of a triangle ABC touches the sides BC, AC, AB in the points D, E, F respectively. Two different points K and L are chosen in ω such that $\angle CKE + \angle BKF = \angle CLE +$

$\angle BLF = 180^\circ$. Prove that the line KL is in the same distance to the point $D, E,$ and F.

8 Let n be a positive integer, such that $125n+22$ is a power of 3. Prove that $125n+29$ has a prime factor greater than 100.