Romania Team Selection Test 2021

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–	Day 1
1	Let $k>1$ be a positive integer. A set S is called *good* if there exists a colouring of the positive integers with k colours, such that no element from S can be written as the sum of two distinct positive integers having the same colour. Find the greatest positive integer t (in terms of k) for which the set $$S=\{a+1,a+2,\ldots,a+t\}$$ is good, for any positive integer a.
2	For any positive integer $n>1$, let $p(n)$ be the greatest prime factor of n. Find all the triplets of distinct positive integers (x,y,z) which satisfy the following properties: x,y and z form an arithmetic progression, and $p(xyz)\leq 3$.
3	The external bisectors of the angles of the convex quadrilateral $ABCD$ intersect each other in E,F,G and H such that $A\in EH$, $B\in EF$, $C\in FG$, $D\in GH$. We know that the perpendiculars from E to AB, from F to BC and from G to CD are concurrent. Prove that $ABCD$ is cyclic.
4	Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ which satisfy the following relationship for all real numbers x and y $$f(xf(y)-f(x))=2f(x)+xy.$$
–	Day 2
1	Find all pairs (m,n) of positive odd integers, such that $n\mid 3m+1$ and $m\mid n^2+3$.
2	Consider the set $M=\{1,2,3,...,2020\}$. Find the smallest positive integer k such that for any subset A of M with k elements, there exist 3 distinct numbers a,b,c from M such that $a+b, b+c$ and $c+a$ are all in A.
3	Let $\mathcal{P}$ be a convex quadrilateral. Consider a point X inside $\mathcal{P}$. Let M,N,P,Q be the projections of X on the sides of $\mathcal{P}$. We know that M,N,P,Q all sit on a circle of center L. Let J and K be the midpoints of the diagonals of $\mathcal{P}$. Prove that J,K and L lie on a line.
–	Day 3

1 Consider a fixed triangle ABC such that $AB = AC$. Let M be the midpoint of BC. Let P be a variable point inside $\triangle ABC$, such that $\angle PBC = \angle PCA$. Prove that the sum of the measures of $\angle BPM$ and $\angle APC$ is constant.

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2 Let $N \geq 4$ be a fixed positive integer. Two players, A and B are forming an ordered set $\{x_1, x_2, ...\}$, adding elements alternatively. A chooses x_1 to be 1 or -1, then B chooses x_2 to be 2 or -2, then A chooses x_3 to be 3 or -3, and so on. (at the k^{th} step, the chosen number must always be k or $-k$)

The winner is the first player to make the sequence sum up to a multiple of N. Depending on N, find out, with proof, which player has a winning strategy.

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3 Let α be a real number in the interval $(0, 1)$. Prove that there exists a sequence $(\varepsilon_n)_{n\geq 1}$ where each term is either 0 or 1 such that the sequence $(s_n)_{n\geq 1}$

$$s_n = \frac{\varepsilon_1}{n(n+1)} + \frac{\varepsilon_2}{(n+1)(n+2)} + ... + \frac{\varepsilon_n}{(2n-1)2n}$$

verifies the inequality

$$0 \leq \alpha - 2ns_n \leq \frac{2}{n+1}$$

for any $n \geq 2$.
