Round 4

www.artofproblemsolving.com/community/c2005112

by jasperE3

– Day 1

Problem 1 Through a random point C_1 from the edge DC of the regular tetrahedron $ABCD$ is drawn a plane, parallel to the plane ABC. The plane constructed intersects the edges DA and DB at the points A_1, B_1 respectively. Let the point H is the midpoint of the altitude through the vertex D of the tetrahedron $DA_1B_1C_1$ and M is the center of gravity (barycenter) of the triangle ABC_1. Prove that the measure of the angle HMC doesnt depend on the position of the point C_1. *(Ivan Tonov)*

Problem 2 Prove that there exists 1904-element subset of the set $\{1, 2, \dots, 1992\}$, which doesnt contain an arithmetic progression consisting of 41 terms. *(Ivan Tonov)*

Problem 3 Let m and n are fixed natural numbers and Oxy is a coordinate system in the plane. Find the total count of all possible situations of $n+m-1$ points $P_1(x_1, y_1), P_2(x_2, y_2), \dots, P_{n+m-1}(x_{n+m-1}, y_{n+m-1})$ in the plane for which the following conditions are satisfied:

(i) The numbers x_i and y_i $(i = 1, 2, \dots, n+m-1)$ are integers and $1 \le x_i \le n, 1 \le y_i \le m$.
(ii) Every one of the numbers $1, 2, \dots, n$ can be found in the sequence $x_1, x_2, \dots, x_{n+m-1}$ and every one of the numbers $1, 2, \dots, m$ can be found in the sequence $y_1, y_2, \dots, y_{n+m-1}$.
(iii) For every $i = 1, 2, \dots, n+m-2$ the line P_iP_{i+1} is parallel to one of the coordinate axes.
(Ivan Gochev, Hristo Minchev)

– Day 2

Problem 4 Let p be a prime number in the form $p = 4k + 3$. Prove that if the numbers x_0, y_0, z_0, t_0 are solutions of the equation $x^{2p} + y^{2p} + z^{2p} = t^{2p}$, then at least one of them is divisible by p. *(Plamen Koshlukov)*

Problem 5 Points D, E, F are midpoints of the sides AB, BC, CA of triangle ABC. Angle bisectors of the angles BDC and ADC intersect the lines BC and AC respectively at the points M and N, and the line MN intersects the line CD at the point O. Let the lines EO and FO intersect respectively the lines AC and BC at the points P and Q. Prove that $CD = PQ$. *(Plamen Koshlukov)*

Problem 6 There are given one black box and n white boxes (n is a random natural number). White boxes are numbered with the numbers $1, 2, \dots, n$. In them are put n balls. It is allowed the following rearrangement of the balls: if in the box with number k there are exactly k balls, that box is made empty - one of the balls is put in the black box and the other $k-1$ balls are put in the boxes with numbers: $1, 2, \ldots, k-1$. *(Ivan Tonov)*
