Iran Team Selection Test 2021

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by Tintarn, WashYourWish, Mr.C

– First exam, Day 1

1 In acute scalene triangle ABC the external angle bisector of $\angle BAC$ meet BC at point X.Lines l_b and l_c which tangents of B and C with respect to (ABC).The line pass through X intersects l_b and l_c at points Y and Z respectively. Suppose $(AYB) \cap (AZC) = N$ and $l_b \cap l_c = D$. Show that ND is angle bisector of $\angle YNZ$.
Proposed by *Alireza Haghi*

2 In the simple and connected graph G let x_i be the number of vertices with degree i. Let $d > 3$ be the biggest degree in the graph G. Prove that if :

$$x_d \geq x_{d-1} + 2x_{d-2} + ... + (d-1)x_1$$

Then there exists a vertex with degree d such that after removing that vertex the graph G is still connected.

Proposed by *Ali Mirzaie*

3 There exist 4 positive integers a, b, c, d such that $abcd \neq 1$ and each pair of them have a GCD of 1. Two functions $f, g : \mathbb{N} \to \{0, 1\}$ are multiplicative functions such that for each positive integer n we have :

$$f(an + b) = g(cn + d)$$

Prove that at least one of the followings hold. $i)$ for each positive integer n we have $f(an+b) = g(cn + d) = 0$ $ii)$ There exists a positive integer k such that for all n where $(n, k) = 1$ we have $g(n) = f(n) = 1$

(Function f is multiplicative if for any natural numbers a, b we have $f(ab) = f(a)f(b)$)

Proposed by *Navid Safaii*

– First exam, Day 2

4 Assume $\Omega(n), \omega(n)$ be the biggest and smallest prime factors of n respectively . Alireza and Amin decided to play a game. First Alireza chooses 1400 polynomials with integer coefficients. Now Amin chooses 700 of them, the set of polynomials of Alireza and Amin are B, A respectively . Amin wins if for all n we have :

$$\max_{P \in A}(\Omega(P(n))) \geq \min_{P \in B}(\omega(P(n)))$$

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Who has the winning strategy.

Proposed by *Alireza Haghi*

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5 Call a triple of numbers **Nice** if one of them is the average of the other two. Assume that we have $2k+1$ distinct real numbers with k^2 **Nice** triples. Prove that these numbers can be devided into two arithmetic progressions with equal ratios

Proposed by *Morteza Saghafian*

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6 Point D is chosen on the Euler line of triangle ABC and it is inside of the triangle. Points E, F are were the line BD, CD intersect with AC, AB respectively. Point X is on the line AD such that $\angle EXF = 180 - \angle A$, also A, X are on the same side of EF. If P is the second intersection of circumcircles of CXF, BXE then prove the lines XP, EF meet on the altitude of A

Proposed by *Alireza Danaie*

– Second Exam, Day 1

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1 Natural numbers are placed in an infinite grid. Such that the number in each cell is equal to the number of its adjacent cells having the same number. Find the most distinct numbers this infinite grid can have.

(Two cells of the grid are adjacent if they have a common vertex)

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2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any two positive integers m, n we have :

$$f(n) + 1400m^2 | n^2 + f(f(m))$$

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3 Prove there exist two relatively prime polynomials $P(x), Q(x)$ having integer coefficients and a real number $u > 0$ such that if for positive integers a, b, c, d we have:

$$|\frac{a}{c} - 1|^{2021} \le \frac{u}{|d||c|^{1010}}$$

$$|(\frac{a}{c})^{2020} - \frac{b}{d}| \le \frac{u}{|d||c|^{1010}}$$

Then we have :

$$bP(\frac{a}{c}) = dQ(\frac{a}{c})$$

(Two polynomials are relatively prime if they don't have a common root)

Proposed by *Navid Safaii* and *Alireza Haghi*

– Second Exam, Day 2

4 Find all functions $f:\mathbb{N}\to\mathbb{R}$ such that for all triples a,b,c of positive integers the following holds :

$$f(ac)+f(bc)-f(c)f(ab)\geq 1$$

Proposed by *Mojtaba Zare*

5 Point X is chosen inside the non trapezoid quadrilateral $ABCD$ such that $\angle AXD+\angle BXC=180$.
Suppose the angle bisector of $\angle ABX$ meets the D-altitude of triangle ADX in K, and the angle bisector of $\angle DCX$ meets the A-altitude of triangle ADX in L.We know $BK\perp CX$ and $CL\perp BX$. If the circumcenter of ADX is on the line KL prove that $KL\perp AD$.

Proposed by *Alireza Dadgarnia*

6 Prove that we can color every subset with n element of a set with $3n$ elements with 8 colors . In such a way that there are no 3 subsets A,B,C with the same color where :

$$|A\cap B|\leq 1, |A\cap C|\leq 1, |B\cap C|\leq 1$$

Proposed by *Morteza Saghafian* and *Amir Jafari*

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