Bulgaria National Olympiad 2021

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– Day 1

1 A city has 4 horizontal and $n \geq 3$ vertical boulevards which intersect at $4n$ crossroads. The crossroads divide every horizontal boulevard into $n-1$ streets and every vertical boulevard into 3 streets. The mayor of the city decides to close the minimum possible number of crossroads so that the city doesn't have a closed path(this means that starting from any street and going only through open crossroads without turning back you can't return to the same street). $a)$Prove that exactly n crossroads are closed. $b)$Prove that if from any street you can go to any other street and none of the 4 corner crossroads are closed then exactly 3 crossroads on the border are closed(A crossroad is on the border if it lies either on the first or fourth horizontal boulevard, or on the first or the n-th vertical boulevard).

2 A point T is given on the altitude through point C in the acute triangle ABC with circumcenter O, such that $\measuredangle TBA = \measuredangle ACB$. If the line CO intersects side AB at point K, prove that the perpendicular bisector of AB, the altitude through A and the segment KT are concurrent.

3 Find all $f: R^+ \rightarrow R^+$ such that $f(f(x)+y)f(x)=f(xy+1)\ \ \forall x,y \in R^+$

@below: `https://artofproblemsolving.com/community/c6h2254883_2020_imoc_problems`

Feel free to start individual threads for the problems as usual

– Day 2

4 Two infinite arithmetic sequences with positive integers are given:

$$a_1<a_2<a_3<\cdots ; b_1<b_2<b_3<\cdots$$

It is known that there are infinitely many pairs of positive integers (i,j) for which $i\leq j\leq i+2021$ and a_i divides b_j. Prove that for every positive integer i there exists a positive integer j such that a_i divides b_j.

5 Does there exist a set S of 100 points in a plane such that the center of mass of any 10 points in S is also a point in S?

6 Point S is the midpoint of arc ACB of the circumscribed circle k around triangle ABC with $AC>BC$. Let I be the incenter of triangle ABC. Line SI intersects k again at point T. Let D

be the reflection of I across T and M be the midpoint of side AB. Line IM intersects the line through D, parallel to AB, at point E. Prove that $AE = BD$.
