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TST-2 TST-2

1 Let x, y, z be positive real numbers such that $x^2+y^2+z^2=3$. Prove that

$$xyz(x+y+z)+2021 \geqslant 2024xyz$$

2 Find all pairs of natural numbers (α, β) for which, if δ is the greatest common divisor of α, β, and $\mathit{\Delta}$ is the least common multiple of α, β, then

$$\delta+\Delta=4(\alpha+\beta)+2021$$

3 Let $AB\mathit{\Gamma}\mathit{\Delta}$ be a rhombus.

(a) Prove that you can construct a circle (c) which is inscribed in the rhombus and is tangent to its sides.
(b) The points $\mathit{\Theta}, H, K, I$ are on the sides $\mathit{\Delta}\mathit{\Gamma}, B\mathit{\Gamma}, AB, A\mathit{\Delta}$ of the rhombus respectively, such that the line segments KH and $I\mathit{\Theta}$ are tangent on the circle (c). Prove that the quadrilateral defined from the points $\mathit{\Theta}, H, K, I$ is a trapezium.

4 We colour every square of a 4×19 chess board with one of the colours red, green and blue. Prove that however this colouring is done, we can always find two horizontal rows and two vertical columns such that the 4 squares on the intersections of these lines all have the same colour.

TST-3 TST-3

1 Find all positive integers n, such that the number

$$\frac{n^{2021}+101}{n^2+n+1}$$

is an integer.

2 Let x, y be real numbers with $x \geqslant \sqrt{2021}$ such that

$$\sqrt[3]{x+\sqrt{2021}}+\sqrt[3]{x-\sqrt{2021}}=\sqrt[3]{y}$$

Determine the set of all possible values of y/x.

3 George plays the following game: At every step he can replace a triple of integers (x, y, z) which is written on the blackboard, with any of the following triples:

(i) (x, z, y)
(ii) $(-x, y, z)$
(iii) $(x + y, y, 2x + y + z)$
(iv) $(x - y, y, y + z - 2x)$

Initially, the triple $(1, 1, 1)$ is written on the blackboard. Determine whether George can, with a sequence of allowed steps, end up at the triple $(2021, 2019, 2023)$, fully justifying your answer.

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4 Let $\triangle AB\Gamma$ be an acute-angled triangle with $AB < A\Gamma$, and let O be the center of the circumcircle of the triangle. On the sides AB and $A\Gamma$ we select points T and P respectively such that $OT = OP$. Let M, K and Λ be the midpoints of PT, PB and ΓT respectively. Prove that $\angle TMK = \angle M\Lambda K$.
