The problems from the 28th Macedonian Mathematical Olympiad

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Problem 1 Let $(a_n)_{n=1}^{+\infty}$ be a sequence defined recursively as follows: $a_1=1$ and

$$a_{n+1} = 1 + \sum_{k=1}^{n} k a_k$$

For every $n>1$, prove that $\sqrt[n]{a_n} < \frac{n+1}{2}$.

Problem 2 In the City of Islands there are 2021 islands connected by bridges. For any given pair of islands A and B, one can go from island A to island B using the bridges. Moreover, for any four islands A_1, A_2, A_3 and A_4: if there is a bridge from A_i to A_{i+1} for each $i \in \{1,2,3\}$, then there is a bridge between A_j and A_k for some $j,k \in \{1,2,3,4\}$ with $|j-k|=2$. Show that there is at least one island which is connected to any other island by a bridge.

Problem 3 Let $ABCD$ be a trapezoid with $AD \parallel BC$ and $\angle BCD < \angle ABC < 90^\circ$. Let E be the intersection point of the diagonals AC and BD. The circumcircle ω of $\triangle BEC$ intersects the segment CD at X. The lines AX and BC intersect at Y, while the lines BX and AD intersect at Z. Prove that the line EZ is tangent to ω iff the line BE is tangent to the circumcircle of $\triangle BXY$.

Problem 4 For a fixed positive integer $n \geq 3$ we are given a $n \times n$ board with all unit squares initially white. We define a *floating plus* as a 5-tuple (M,L,R,A,B) of unit squares such that L is in the same row and left of M, R is in the same row and right of M, A is in the same column and above M and B is in the same column and below M. It is possible for M to form a floating plus with unit squares that are not next to it. Find the largest positive integer k (depending on n) such that we can color some k squares black in such a way that there is no black colored floating plus.

Proposed by Nikola Velov

Problem 5 Let $(x_n)_{n=1}^{+\infty}$ be a sequence defined recursively with $x_{n+1} = x_n(x_n-2)$ and $x_1 = \frac{7}{2}$. Let $x_{2021} = \frac{a}{b}$, where $a,b \in \mathbb{N}$ are coprime. Show that if p is a prime divisor of a, then either $3|p-1$ or $p=3$.

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