Turkey Team Selection Test 2021

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Day 1 19 May 2021

1 Let n be a positive integer. Prove that

$$\frac{20 \cdot 5^n - 2}{3^n + 47}$$

is not an integer.

2 In a school with some students, for any three student, there exists at least one student who are friends with all these three students.(Friendships are mutual) For any friends A and B, any two of their common friends are also friends with each other. It's not possible to partition these students into two groups, such that every student in each group are friends with all the students in the other gruop. Prove that any two people who aren't friends with each other, has the same number of common friends.(Each person is a friend of him/herself.)

3 A point D is taken on the arc BC of the circumcircle of triangle ABC which does not contain A. A point E is taken at the intersection of the interior region of the triangles ABC and ADC such that $m(\widehat{ABE}) = m(\widehat{BCE})$. Let the circumcircle of the triangle ADE meets the line AB for the second time at K. Let L be the intersection of the lines EK and BC, M be the intersection of the lines EC and AD, N be the intersection of the lines BM and DL. Prove that

$$m(\widehat{NEL}) = m(\widehat{NDE})$$

Day 2 20 May 2021

4 In a fish shop with 28 kinds of fish, there are 28 fish sellers. In every seller, there exists only one type of each fish kind, depending on where it comes, Mediterranean or Black Sea. Each of the k people gets exactly one fish from each seller and exactly one fish of each kind. For any two people, there exists a fish kind which they have different types of it (one Mediterranean, one Black Sea). What is the maximum possible number of k?

5 In a non isoceles triangle ABC, let the perpendicular bisector of $[BC]$ intersect (ABC) at M and N respectively. Let the midpoints of $[AM]$ and $[AN]$ be K and L respectively. Let (ABK) and (ABL) intersect AC again at D and E respectively, let (ACK) and (ACL) intersect AB again at F and G respectively.
Prove that the lines DF, EG and MN are concurrent.

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6 For which positive integers n, one can find real numbers $x_1, x_2, \cdots, x_n$ such that

$$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{(x_1 + 2x_2 + \cdots + nx_n)^2} = \frac{27}{4n(n+1)(2n+1)}$$

and $i \le x_i \le 2i$ for all $i = 1, 2, \cdots, n$?

Day 3 21 May 2021

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7 Given a triangle ABC with the circumcircle ω and incenter I. Let the line pass through the point I and the intersection of exterior angle bisector of A and ω meets the circumcircle of IBC at T_A for the second time. Define T_B and T_C similarly. Prove that the radius of the circumcircle of the triangle $T_AT_BT_C$ is twice the radius of ω.

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8 Let c be a real number. For all x and y real numbers we have,

$$f(x - f(y)) = f(x - y) + c(f(x) - f(y))$$

and $f(x)$ is not constant. $a)$ Find all possible values of c. $b)$ Can f be periodic?

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9 For which positive integer couples (k, n), the equality

$$\left|\left\{a \in \mathbb{Z}^+ : 1 \le a \le (nk)!, gcd\left(\binom{a}{k}, n\right) = 1\right\}\right| = \frac{(nk)!}{6}$$

holds?

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