Pan-African 2021

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–   Day 1

1   Let n be an integer greater than 3. A square of side length n is divided by lines parallel to each side into n^2 squares of length 1. Find the number of convex trapezoids which have vertices among the vertices of the n^2 squares of side length 1, have side lengths less than or equal 3 and have area equal to 2
Note: Parallelograms are trapezoids.

2   Let Γ be a circle, P be a point outside it, and A and B the intersection points between Γ and the tangents from P to Γ. Let K be a point on the line AB, distinct from A and B and let T be the second intersection point of Γ and the circumcircle of the triangle PBK. Also, let P' be the reflection of P in point A.
Show that $\angle PBT = \angle P'KA$

3   Let $(a_i)_{i \in \mathbb{N}}$ and $(p_i)_{i \in \mathbb{N}}$ be two sequences of positive integers such that the following conditions hold: • $a_1 \geq 2$. • p_n is the smallest prime divisor of a_n for every integer $n \geq 1$ • $a_{n+1} = a_n + \frac{a_n}{p_n}$ for every integer $n \geq 1$
Prove that there is a positive integer N such that $a_{n+3} = 3a_n$ for every integer $n > N$

–   Day 2

4   Find all integers m and n such that $\frac{m^2+n}{n^2-m}$ and $\frac{n^2+m}{m^2-n}$ are both integers.

5   Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$:

$$(f(x) + y)(f(y) + x) = f(x^2) + f(y^2) + 2f(xy)$$

6   Let $ABCD$ be a trapezoid which is not a parallelogram, such that AD is parallel to BC.
Let $O = BD \cap AC$ and S be the second intersection of the circumcircles of triangles AOB and DOC.
Prove that the circumcircles of triangles ASD and BSC are tangent.