CJMO - Canadian Junior Mathematical Olympiad 2021

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1 Let C_1 and C_2 be two concentric circles with C_1 inside C_2. Let P_1 and P_2 be two points on C_1 that are not diametrically opposite. Extend the segment P_1P_2 past P_2 until it meets the circle C_2 in Q_2. The tangent to C_2 at Q_2 and the tangent to C_1 at P_1 meet in a point X. Draw from X the second tangent to C_2 which meets C_2 at the point Q_1. Show that P_1X bisects angle $Q_1P_1Q_2$.

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2 How many ways are there to permute the first n positive integers such that in the permutation, for each value of $k \le n$, the first k elements of the permutation have distinct remainder mod k?

– those were also CMO problems

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3 Let $ABCD$ be a trapezoid with AB parallel to CD, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB, AC and BD, where A and I are on opposite sides of BD. Let J be the center of the circle tangent to lines CD, AC and BD, where D and J are on opposite sides of AC. Prove that $|IC| = |JB|$.

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4 Let $n \ge 2$ be some fixed positive integer and suppose that $a_1, a_2, \dots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}$$

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5 A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\left(f(f(x)), f(x+y)\right) = \gcd(x, y)$$

for all pairs of positive integers x and y.

Find all positive integers m such that $f(m) = m$ for all Canadian functions f.
