Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 2021

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1a A $3n$-table is a table with three rows and n columns containing all the numbers $1, 2, , 3n$. Such a table is called *tidy* if the n numbers in the first row appear in ascending order from left to right, and the three numbers in each column appear in ascending order from top to bottom. How many tidy $3n$-tables exist?

1b Pl has more chickens than he can manage to keep track of. Therefore, he keeps an index card for each chicken. He keeps the cards in ten boxes, each of which has room for 2021 cards. Unfortunately, Pl is quite disorganized, so he may lose some of his boxes. Therefore, he makes several copies of each card and distributes them among different boxes, so that even if he can only find seven boxes, no matter which seven, these seven boxes taken together will contain at least one card for each of his chickens.
What is the largest number of chickens Pl can keep track of using this system?

2a Show that for all $n \geq 3$ there are n different positive integers $x_1, x_2, ..., x_n$ such that

$$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} = 1.$$

2b If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are real numbers satisfying $a_1^2 + \cdots + a_n^2 \leq 1$ and $b_1^2 + \cdots + b_n^2 \leq 1$, show that:

$$(1 - (a_1^2 + \cdots + a_n^2))(1 - (b_1^2 + \cdots + b_n^2)) \leq (1 - (a_1b_1 + \cdots + a_nb_n))^2$$

3a For which integers $0 \leq k \leq 9$ do there exist positive integers m and n so that the number $3^m + 3^n + k$ is a perfect square?

3b We say that a set S of natural numbers is *synchronous* provided that the digits of a^2 are the same (in occurence and numbers, if differently ordered) for all numbers a in S. For example, $\{13, 14, 31\}$ is synchronous, since we find $\{13^2, 14^2, 31^2\} = \{169, 196, 961\}$. But $\{119, 121\}$ is not synchronous, for even though $119^2 = 14161$ and $121^2 = 14641$ have the same digits, they occur in different numbers. Show that there exists a synchronous set containing 2021 different natural numbers.

4a A tetrahedron $ABCD$ satisfies $\angle BAC = \angle CAD = \angle DAB = 90^o$. Show that the areas of its faces satisfy the equation $area(BAC)^2 + area(CAD)^2 + area(DAB)^2 = area(BCD)^2$.
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4b The tangent at C to the circumcircle of triangle ABC intersects the line through A and B in a point D. Two distinct points E and F on the line through B and C satisfy $|BE| = |BF| = \frac{||CD|^2-|BD|^2|}{|BC|}$. Show that either $|ED| = |CD|$ or $|FD| = |CD|$.