8 May 2021

www.artofproblemsolving.com/community/c2015210

by parmenides51, circlethm

–	Paper 1
1	Let $N = 15! = 15 \cdot 14 \cdot 13...3 \cdot 2 \cdot 1$. Prove that N can be written as a product of nine different integers all between 16 and 30 inclusive.
2	An isosceles triangle ABC is inscribed in a circle with $\angle ACB = 90^o$ and EF is a chord of the circle such that neither E nor F coincide with C. Lines CE and CF meet AB at D and G respectively. Prove that $\lvert CE\rvert \cdot \lvert DG\rvert = \lvert EF\rvert \cdot \lvert CG\rvert$.
3	For each integer $n \ge 100$ we define $T(n)$ to be the number obtained from n by moving the two leading digits to the end. For example, $T(12345) = 34512$ and $T(100) = 10$. Find all integers $n \ge 100$ for which $n + T(n) = 10n$.
4	You have a 3×2021 chessboard from which one corner square has been removed. You also have a set of 3031 identical dominoes, each of which can cover two adjacent chessboard squares. Let m be the number of ways in which the chessboard can be covered with the dominoes, without gaps or overlaps. What is the remainder when m is divided by 19?
5	The function $g : [0, \infty) \to [0, \infty)$ satisfies the functional equation: $g(g(x)) = \frac{3x}{x+3}$, for all $x \ge 0$. You are also told that for $2 \le x \le 3$: $g(x) = \frac{x+1}{2}$. (a) Find $g(2021)$. (b) Find $g(1/2021)$.
–	Paper 2
6	A sequence whose first term is positive has the property that any given term is the area of an equilateral triangle whose perimeter is the preceding term. If the first three terms form an arithmetic progression, determine all possible values of the first term.
7	Each square of an $n \times n$ grid is coloured either blue or red, where n is a positive integer. There are k blue cells in the grid. Pat adds the sum of the squares of the numbers of blue cells in each row to the sum of the squares of the numbers of blue cells in each column to form S_B. He then performs the same calculation on the red cells to compute S_R. If $S_B - S_R = 50$, determine (with proof) all possible values of k.

8 A point C lies on a line segment AB between A and B and circles are drawn having AC and CB as diameters. A common tangent to both circles touches the circle with AC as diameter at $P \neq C$ and the circle with CB as diameter at $Q \neq C$.
Prove that AP, BQ and the common tangent to both circles at C all meet at a single point which lies on the circumference of the circle with AB as diameter.

---

9 Suppose the real numbers a, A, b, B satisfy the inequalities:

$$|A - 3a| \leq 1 - a \ \ , \ \ |B - 3b| \leq 1 - b$$

, and a, b are positive. Prove that

$$\left|\frac{AB}{3} - 3ab\right| - 3ab \leq 1 - ab.$$

---

10 Let $P_1, P_2, \ldots, P_{2021}$ be 2021 points in the quarter plane $\{(x, y) : x \geq 0, y \geq 0\}$. The centroid of these 2021 points lies at the point $(1, 1)$.

Show that there are two distinct points P_i, P_j such that the distance from P_i to P_j is no more than $\sqrt{2}/20$.
