Round 4

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– Day 1

Problem 1 Let M be a point on the altitude CD of an acute-angled triangle ABC, and K and L the orthogonal projections of M on AC and BC. Suppose that the incenter and circumcenter of the triangle lie on the segment KL.

(a) Prove that $CD=R+r$, where R and r are the circumradius and inradius, respectively.
(b) Find the minimum value of the ratio $CM:CD$.

Problem 2 Let K be a cube with edge n, where $n>2$ is an even integer. Cube K is divided into n^3 unit cubes. We call any set of n^2 unit cubes lying on the same horizontal or vertical level a layer. We dispose of $\frac{n^3}{4}$ colors, in each of which we paint exactly 4 unit cubes. Prove that we can always select n unit cubes of distinct colors, no two of which lie on the same layer.

Problem 3 Prove that for every prime number $p\ge5$,

(a) p^3 divides $\binom{2p}{p}-2$;
(b) p^3 divides $\binom{kp}{p}-k$ for every natural number k.

– Day 2

Problem 4 Let $f(x)$ be a polynomial of degree n with real coefficients, having n (not necessarily distinct) real roots. Prove that for all real x,

$$f(x)f''(x)\le f'(x)^2.$$

Problem 5 On a unit circle with center O, AB is an arc with the central angle $\alpha<90^\circ$. Point H is the foot of the perpendicular from A to OB, T is a point on arc AB, and l is the tangent to the circle at T. The line l and the angle AHB form a triangle Δ.

(a) Prove that the area of Δ is minimal when T is the midpoint of arc AB.
(b) Prove that if S_α is the minimal area of Δ then the function $\frac{S_\alpha}{\alpha}$ has a limit when $\alpha\to0$ and find this limit.

Problem 6 White and black checkers are put on the squares of an $n\times n$ chessboard $(n\ge2)$ according to the following rule. Initially, a black checker is put on an arbitrary square. In every consequent step, a white checker is put on a free square, whereby all checkers on the squares neighboring by side are replaced by checkers of the opposite colors. This process is continued until there is a checker on every square. Prove that in the final configuration there is at least one black checker.