National Olympiad Second Round 2015

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by Eray

Day 1 December 5th

1 m and n are positive integers. If the number

$$k = \frac{(m+n)^2}{4m(m-n)^2+4}$$

is an integer, prove that k is a perfect square.

2 x, y and z are real numbers where the sum of any two among them is not 1. Show that,

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}$$

Find all triples (x,y,z) of real numbers satisfying the equality case.

3 n points are given on a plane where $n \geq 4$. All pairs of points are connected with a segment. Find the maximal number of segments which don't intersect with any other segments in their interior.

Day 2 December 6th

4 In an exhibition where 2015 paintings are shown, every participant picks a pair of paintings and writes it on the board. Then, Fake Artist (F.A.) chooses some of the pairs on the board, and marks one of the paintings in all of these pairs as "better". And then, Artist's Assistant (A.A.) comes and in his every move, he can mark A better then C in the pair (A,C) on the board if for a painting B, A is marked as better than B and B is marked as better than C on the board. Find the minimum possible value of k such that, for any pairs of paintings on the board, F.A can compare k pairs of paintings making it possible for A.A to compare all of the remaining pairs of paintings.

P.S: A.A can decide $A_1 > A_n$ if there is a sequence $A_1 > A_2 > A_3 > \cdots > A_{n-1} > A_n$ where $X > Y$ means painting X is better than painting Y.

5 In a cyclic quadrilateral $ABCD$ whose largest interior angle is D, lines BC and AD intersect at point E, while lines AB and CD intersect at point F. A point P is taken in the interior of quadrilateral $ABCD$ for which $\angle EPD = \angle FPD = \angle BAD$. O is the circumcenter of quadrilateral $ABCD$. Line FO intersects the lines AD, EP, BC at X, Q, Y, respectively. If $\angle DQX = \angle CQY$, show that $\angle AEB = 90°$.

6 Find all positive integers n such that for any positive integer a relatively prime to n, $2n^2 \mid a^n - 1$.
