Serbia National Math Olympiad

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–	Day 1
1	In a scalene triangle ABC, α and β respectively denote the interior angles at vertixes A and B. The bisectors of these two angles meet the opposite sides of the triangle at points D and E, respectively. Prove that the acute angle between the lines DE and AB does not exceed $\frac{\lvert\alpha-\beta\rvert}{3}$. *Proposed by Dusan Djukic*
2	Find the smallest natural number which is a multiple of 2009 and whose sum of (decimal) digits equals 2009 *Proposed by Milos Milosavljevic*
3	Determine the largest positive integer n for which there exist pairwise different sets $\mathbb{S}_1, ..., \mathbb{S}_n$ with the following properties: $1)$ $\lvert\mathbb{S}_i \cup \mathbb{S}_j\rvert \leq 2004$ for any two indices $1 \leq i, j \leq n$, and $2)$ $\mathbb{S}_i \cup \mathbb{S}_j \cup \mathbb{S}_k = \{1, 2, ..., 2008\}$ for any $1 \leq i < j < k \leq n$ *Proposed by Ivan Matic*
–	Day 2
4	Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which $$k \mid 2(a_1 + \cdots + a_k), \text{ for all } 1 \leq k \leq n.$$ Find the number of elements of the set A_n. *Proposed by Vidan Govedarica, Serbia*
5	Let x, y, z be arbitrary positive numbers such that $xy + yz + zx = x + y + z$. Prove that $$\frac{1}{x^2+y+1} + \frac{1}{y^2+z+1} + \frac{1}{z^2+x+1} \leq 1$$. When does equality occur? *Proposed by Marko Radovanovic*
6	Triangle ABC has incircle w centered as S that touches the sides BC,CA and AB at P,Q and R respectively. AB isn't equal AC, the lines QR and BC intersects at point M, the circle that passes through points B and C touches the circle w at point N, circumcircle of triangle MNP intersects with line AP at L (L isn't equal to P). Then prove that S,L and M lie on the same line