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1 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at $A_4 (\neq A_2)$. Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

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2 Let $a_1, a_2, \cdots, a_k$ be numbers such that $a_i \in \{0, 1, 2, 3\} (i = 1, 2, \cdots, k)$. Let $z = (x_k, x_{k-1}, \cdots, x_1)_4$ be a base 4 expansion of $z \in \{0, 1, 2, \cdots, 4^k - 1\}$. Define A as follows:

$$A = \{z | p(z) = z, z = 0, 1, \cdots, 4^k - 1\}$$

where

$$p(z) = \sum_{i=1}^{k} a_i x_i 4^{i-1}.$$

Prove that the number of elements in X is a power of 2.

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3 Find all $a, b, c \in \mathbb{Z}$, $c \geq 0$ such that $a^n + 2^n | b^n + c$ for all positive integers n where $2ab$ is non-square.

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4 Positive integers 1 to 9 are written in each square of a 3×3 table. Let us define an operation as follows: Take an arbitrary row or column and replace these numbers a, b, c with either non-negative numbers $a - x, b - x, c + x$ or $a + x, b - x, c - x$, where x is a positive number and can vary in each operation.
(1) Does there exist a series of operations such that all 9 numbers turn out to be equal from the following initial arrangement a)? b)?

$$\begin{array}{cccc} & 1 & 2 & 3 \\ a) & 4 & 5 & 6\ ) \\ & 7 & 8 & 9 \end{array}$$

$$\begin{array}{cccc} & 2 & 8 & 5 \\ b) & 9 & 3 & 4\ ) \\ & 6 & 7 & 1 \end{array}$$

(2) Determine the maximum value which all 9 numbers turn out to be equal to after some steps.

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5 The incircle ω of a quadrilateral $ABCD$ touches AB, BC, CD, DA at E, F, G, H, respectively. Choose an arbitrary point X on the segment AC inside ω. The segments XB, XD meet ω at I, J respectively. Prove that FJ, IG, AC are concurrent.

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6 Show that $x^3 + x + a^2 = y^2$ has at least one pair of positive integer solution (x, y) for each positive integer a.