Czech And Slovak Mathematical Olympiad, Round III, Category A 2021

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by parmenides51

1 A fraction with 1010 squares in the numerator and 1011 squares in the denominator serves as a game board for a two player game.

$$\frac{\square + \square + ... + \square}{\square + \square + ... + \square + \square}$$

Players take turns in moves. In each turn, the player chooses one of the numbers $1, 2, ..., 2021$ and inserts it in any empty field. Each number can only be used once. The starting player wins if the value of the fraction after all the fields is filled differs from number 1 by less than 10^{-6}. Otherwise, the other player wins. Decide which of the players has a winning strategy.

(Pavel alom)

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2 Let I denote the center of the circle inscribed in the right triangle ABC with right angle at the vertex A. Next, denote by M and N the midpoints of the lines AB and BI. Prove that the line CI is tangent to the circumscribed circle of triangle BMN.

(Patrik Bak, Josef Tkadlec)

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3 The different non-zero real numbers a, b, c satisfy the set equality $\{a+b, b+c, c+a\} = \{ab, bc, ca\}$.
Prove that the set equality $\{a, b, c\} = \{a^2 - 2, b^2 - 2, c^2 - 2\}$ also holds.
.
(Josef Tkadlec)

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4 Find all natural numbers n for which equality holds $n + d(n) + d(d(n)) + ... = 2021$, where $d(0) = d(1) = 0$ and for $k > 1$, $d(k)$ is the *superdivisor* of the number k (i.e. its largest divisor of d with property $d < k$).

(Tom Brta)

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5 We call a string of characters *neat* when it has an even length and its first half is identical to the other half (eg. *abab*). We call a string *nice* if it can be split on several neat strings (e.g. *abcabcdedef* to *abcabc*, *dede*, and *ff*). By string *reduction* we call an operation in which we wipe two identical adjacent characters from the string (e.g. the string *abbac* can be reduced to *aac* and further to *c*). Prove any string containing each of its characters in even numbers can be obtained by a series of reductions from a suitable nice string.

(Martin Melicher)

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6 An acute triangle ABC is given. Let us denote X for each of its inner points X_a, X_b, X_c its images in axial symmetries sequentially along the lines BC, CA, AB. Prove that all $X_aX_bX_c$ triangles have a common interior point.

(Josef Tkadlec)