Kosovo National Mathematical Olympiad 2012

www.artofproblemsolving.com/community/c2330061

by parmenides51, Com10atorics

–	Grade 9
1	Find the value of $(1+2)(1+2^2)(1+2^4)(1+2^8)...(1+2^{2048})$.
2	If $a>1, b>1$ are the legths of the catheti of an right triangle and c the length of its hypotenuse, prove that $a+b\leq c\sqrt{2}$
3	The integers $a_1,a_2,...,a_{2012}$ are given. Exactly 29 of them are divisible by 3. Prove that the sum $a_1^2+a_2^2+...+a_{2012}^2$ is divisible by 3.
4	Inside of the square $ABCD$ the point P is given such that $\|PA\|:\|PB\|:\|PC\|=1:2:3$. Find $\angle APB$.
5	The following square table is given with seven raws and seven columns: $a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}$ $a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_{27}$ $a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, a_{36}, a_{37}$ $a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, a_{46}, a_{47}$ $a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, a$ $a_{61}, a_{62}, a_{63}, a_{64}, a_{65}, a_{66}, a_{67}$ $a_{71}, a_{72}, a_{73}, a_{74}, a_{75}, a_{76}, a_{77}$ Suppose $a_{ij}\in\{0,1\},\forall i,j\in\{1,...,7\}$. Prove that there exists at least one combination of the numbers 1 and 0 so that the following conditions hold: (i) Each raw and each column has exactly three 1's. $(ii)\sum_{j=1}^{7}a_{lj}a_{ij}=1, \forall l,i\in\{1,...,7\}$ and $l\neq i$.(so for any two distinct raws there is exactly one r so that the both raws have 1 in the r-th place). $(iii)\sum_{i=1}^{7}a_{ij}a_{ik}=1,\forall j,k\in\{1,...,7\}$ and $j\neq k$.(so for any two distinct columns there is exactly one s so that the both columns have 1 in the s-th place).
–	Grade 10
1	Find the two last digits of 2012^{2012}.
2	Let a,b,c be the lengths of the sides of a triangle. Prove that, $\left\|\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\frac{b}{a}-\frac{c}{b}-\frac{a}{c}\right\|<1$
3	Let $n\not\equiv 2 \pmod 3$. Is $\sqrt{\lfloor n+\frac{2n}{3}\rfloor+7}, \forall n\in\mathbb{N}$, a natural number?
4	The right triangle ABC with a right angle at C. From all the rectangles CA_1MB_1, where $A_1\in BC, M\in AB$ and $B_1\in AC$ which one has the biggest area?
5	same as 9.5

–	Grade 11
1	If $(x^2-x-1)^n = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + ... + a_1x + a_0$, where $a_i, i \in \{0,1,2,..,2n\}$, find $a_1 + a_3 + ... + a_{2n-1}$ and $a_0 + a_2 + a_4 + ... + a_{2n}$.
2	The bisector of acute angle α of the right triangle ABC splits the side a in two segments with lengths m and n. Find the acute angle β and the lenths of the other two sides by knowing m and n.
3	Prove that for any integer $n \geq 2$ it holds that $\binom{2n}{n} > \frac{4^n}{2n}$.
4	Find the set of solutions to the equation $\log_{\lfloor x \rfloor}(x^2-1) = 2$
5	same as 9.5
–	Grade 12
1	Prove that for all $n \in \mathbb{N}$ we have $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
2	In a sphere S_0 we radius r a cube K_0 has been inscribed. Then in the cube K_0 another sphere S_1 has been inscribed and so on to infinity. Calculate the volume of all spheres created in this way.
3	Solve the recurrence $R_0 = 1, R_n = nR_{n-1} + 2^n \cdot n!$.
4	Let x, y be positive real numbers such that $x+y+xy = 3$. Prove that $x+y \geq 2$. For what values of x and y do we have $x+y = 2$?
5	same as 9.5