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1 Prove that for each real number $r > 2$, there are exactly two or three positive real numbers x satisfying the equation $x^2 = r\lfloor x \rfloor$.

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2 For a polynomial P and a positive integer n, define P_n as the number of positive integer pairs (a, b) such that $a < b \leq n$ and $|P(a)| - |P(b)|$ is divisible by n. Determine all polynomial P with integer coefficients such that $P_n \leq 2021$ for all positive integers n.

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3 Let $ABCD$ be a cyclic convex quadrilateral and Γ be its circumcircle. Let E be the intersection of the diagonals of AC and BD. Let L be the center of the circle tangent to sides AB, BC, and CD, and let M be the midpoint of the arc BC of Γ not containing A and D. Prove that the excenter of triangle BCE opposite E lies on the line LM.

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4 Given a 32×32 table, we put a mouse (facing up) at the bottom left cell and a piece of cheese at several other cells. The mouse then starts moving. It moves forward except that when it reaches a piece of cheese, it eats a part of it, turns right, and continues moving forward. We say that a subset of cells containing cheese is good if, during this process, the mouse tastes each piece of cheese exactly once and then falls off the table. Show that:

(a) No good subset consists of 888 cells.
(b) There exists a good subset consisting of at least 666 cells.

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5 Determine all Functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(a) - b) + bf(2a)$ is a perfect square for all integers a and b.
