USAMTS Problems 2015

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– Round 1

– October 19th

1 Fill in the spaces of the grid below with positive integers so that in each 2×2 square with top left number a, top right number b, bottom left number c, and bottom right number d, either $a+d=b+c$ or $ad=bc$. You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

3	9			
	11		7	2
10				16
15				
20	36			32

2 **2/1/27.** Suppose $a, b,$ and c are distinct positive real numbers such that

$$abc = 1000,$$
$$bc(1-a) + a(b+c) = 110.$$

If $a < 1$, show that $10 < c < 100$.

3 Let P be a convex n-gon in the plane with vertices labeled $V_1, ..., V_n$ in counterclockwise order. A point Q not outside P is called a balancing point of P if, when the triangles the blue and green regions are the same. Suppose P has exactly one balancing point/ Show that the balancing point must be a vertex of P

4 Several players try out for the USAMTS basketball team, and they all have integer heights and weights when measured in centimeters and pounds, respectively. In addition, they all weigh less in pounds than they are tall in centimeters. All of the players weigh at least 190 pounds and are at most 197 centimeters tall, and there is exactly one player with every possible height-weight combination.

The USAMTS wants to field a competitive team, so there are some strict requirements.

- If person P is on the team, then anyone who is at least as tall and at most as heavy as P must also be on the team.
- If person P is on the team, then no one whose weight is the same as Ps height can also be on the team.

Assuming the USAMTS team can have any number of members (including zero), how many different basketball teams can be constructed?

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5 Find all positive integers n that have distinct positive divisors $d_1, d_2, \dots, d_k$, where $k > 1$, that are in arithmetic progression and

$$n = d_1 + d_2 + \cdots + d_k.$$

Note that $d_1, d_2, \dots, d_k$ do not have to be all the divisors of n.

– Round 2

– November 30th

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1 In the grid to the right, the shortest path through unit squares between between the pair of 2's has length 2. Fill in some of the unit squares in the grid so that

(i) exactly half of the squares in each row and column contain a number,

(ii) each of the number 1 through 12 appears exactly twice, and

(iii) for $n = 1, 2, \cdots, 12$, the shortest path between the pair of n's has length exactly n.

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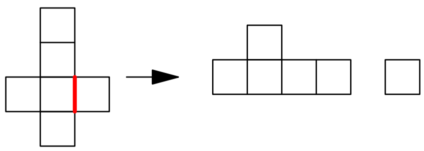
2 A net for a polyhedron is cut along an edge to give two **pieces**. For example, we may cut a cube net along the red edge to form two pieces as shown.



Are there two distinct polyhedra for which this process may result in the same two pairs of pieces? If you think the answer is no, prove that no pair of polyhedra can result in the same two pairs of pieces. If you think the answer is yes, provide an example; a clear example will suffice as a proof.

3 For all positive integers n, show that:

$$\frac{1}{n}\sum_{k=1}^{n}\frac{k\cdot k!\cdot\binom{n}{k}}{n^k}=1$$

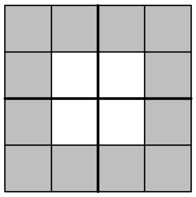
4 Find all polynomials $P(x)$ with integer coefficients such that, for all integers a and b, $P(a+b)-P(b)$ is a multiple of $P(a)$.

5 Let $n>1$ be an even positive integer. An $2n\times 2n$ grid of unit squares is given, and it is partitioned into n^2 contiguous 2×2 blocks of unit squares. A subset S of the unit squares satisfies the following properties:

(i) For any pair of squares A,B in S, there is a sequence of squares in S that starts with A, ends with B, and has any two consecutive elements sharing a side; and

(ii) In each of the 2×2 blocks of squares, at least one of the four squares is in S.

An example for $n=2$ is shown below, with the squares of S shaded and the four 2×2 blocks of squares outlined in bold.



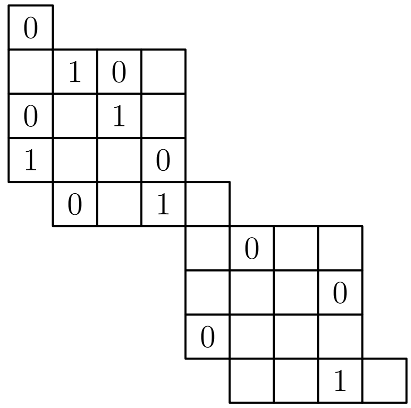
In terms of n, what is the minimum possible number of elements in S?

– Round 3

– January 4th

1 Fill in each space of the grid with either a 0 or a 1 so that all 16 strings of four consecutive numbers across and down are distinct.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable).



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2 Fames is playing a computer game with falling two-dimensional blocks. The playing field is 7 units wide and infinitely tall with a bottom border. Initially the entire field is empty. Each turn, the computer gives Fames a 1×3 solid rectangular piece of three unit squares. Fames must decide whether to orient the piece horizontally or vertically and which column(s) the piece should occupy (3 consecutive columns for horizontal pieces, 1 column for vertical pieces). Once he confirms his choice, the piece is dropped straight down into the playing field in the selected columns, stopping all three of the piece's squares as soon as the piece hits either the bottom of the playing field or any square from another piece. All of the pieces must be contained completely inside the playing field after dropping and cannot partially occupy columns.

If at any time a row of 7 spaces is all filled with squares, Fames scores a point.

Unfortunately, Fames is playing in *invisible mode*, which prevents him from seeing the state of the playing field or how many points he has, and he has already arbitrarily dropped some number of pieces without remembering what he did with them or how many there were.

For partial credit, find a strategy that will allow Fames to eventually earn at least one more point. For full credit, find a strategy for which Fames can correctly announce "I have earned at least one more point" and know that he is correct.

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3 For $n > 1$, let a_n be the number of zeroes that $n!$ ends with when written in base n. Find the maximum value of $\frac{a_n}{n}$.

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4 Let $\triangle ABC$ be a triangle with $AB < AC$. Let the angle bisector of $\angle BAC$ meet BC at D, and let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from B to $\overline{AD}$. Extend $\overline{BP}$ to meet $\overline{AM}$ at Q. Show that $\overline{DQ}$ is parallel to $\overline{AB}$.

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5 Let $a_1, a_2, \ldots, a_{100}$ be a sequence of integers. Initially, $a_1 = 1$, $a_2 = -1$ and the remaining numbers are 0. After every second, we perform the following process on the sequence: for $i = 1, 2, \ldots, 99$, replace a_i with $a_i + a_{i+1}$, and replace a_{100} with $a_{100} + a_1$. (All of this is done simultaneously, so each new term is the sum of two terms of the sequence from before any replacements.) Show that for any integer M, there is some index i and some time t for which $|a_i| > M$ at time t.