2014, Bay Area Mathematical Olympiad 8/12

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– BAMO-8

1 The four bottom corners of a cube are colored red, green, blue, and purple. How many ways are there to color the top four corners of the cube so that every face has four different colored corners? Prove that your answer is correct.

2 There are n holes in a circle. The holes are numbered $1, 2, 3$ and so on to n. In the beginning, there is a peg in every hole except for hole 1. A peg can jump in either direction over one adjacent peg to an empty hole immediately on the other side. After a peg moves, the peg it jumped over is removed. The puzzle will be solved if all pegs disappear except for one. For example, if $n = 4$ the puzzle can be solved in two jumps: peg 3 jumps peg 4 to hole 1, then peg 2 jumps the peg in 1 to hole 4. (See illustration below, in which black circles indicate pegs and white circles are holes.)

`http://i.imgur.com/4gg0a8m.png`

-Can the puzzle be solved for $n = 5$?
-Can the puzzle be solved for $n = 2014$?

In each part (a) and (b) either describe a sequence of moves to solve the puzzle or explain why it is impossible to solve the puzzle.

3 Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won. Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct.

4 Let $\triangle ABC$ be a scalene triangle with the longest side AC. (A *scalene triangle* has sides of different lengths.) Let P and Q be the points on the side AC such that $AP = AB$ and $CQ = CB$. Thus we have a new triangle $\triangle BPQ$ inside $\triangle ABC$. Let k_1 be the circle circumscribed around the triangle $\triangle BPQ$ (that is, the circle passing through the vertices $B, P,$ and Q of the triangle $\triangle BPQ$); and let k_2 be the circle inscribed in triangle $\triangle ABC$ (that is, the circle inside triangle

$\triangle ABC$ that is tangent to the three sides AB, BC, and CA). Prove that the two circles k_1 and k_2 are concentric, that is, they have the same center.

– BAMO-12

1 Same as BAMO-8#3

2 Same as BAMO-8#4

3 Suppose that for two real numbers x and y the following equality is true:

$$(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})=1$$

Find (with proof) the value of $x+y$.

4 Let $F_1, F_2, F_3 \cdots$ be the Fibonacci sequence, the sequence of positive integers satisfying

$$F_1 = F_2 = 1$$

and

$$F_{n+2} = F_{n+1} + F_n$$

for all $n \geq 1$.

Does there exist an $n \geq 1$ such that F_n is divisible by 2014? Prove your answer.

5 A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings. It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n-\sqrt{2k}$ and no more than $n+\sqrt{2k}$ games.