2016, Bay Area Mathematical Olympiad 8/12

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by adihaya

–	February 23rd, 2016	
–	BAMO-8	

1 The diagram below is an example of a *rectangle tiled by squares*:

`http://i.imgur.com/XCPQJgk.png`

Each square has been labeled with its side length. The squares fill the rectangle without overlapping. In a similar way, a rectangle can be tiled by nine squares whose side lengths are $2, 5, 7, 9, 16, 25, 28, 33$, and 36. Sketch one such possible arrangement of those squares. They must fill the rectangle without overlapping. Label each square in your sketch by its side length as in the picture above.

2 A weird calculator has a numerical display and only two buttons, $\boxed{D\sharp}$ and $\boxed{D\flat}$. The first button doubles the displayed number and then adds 1. The second button doubles the displayed number and then subtracts 1. For example, if the display is showing 5, then pressing the $\boxed{D\sharp}$ produces 11. If the display shows 5 and we press $\boxed{D\flat}$, we get 9. If the display shows 5 and we press the sequence $\boxed{D\sharp}$, $\boxed{D\flat}$, $\boxed{D\sharp}$, $\boxed{D\sharp}$, we get a display of 87.

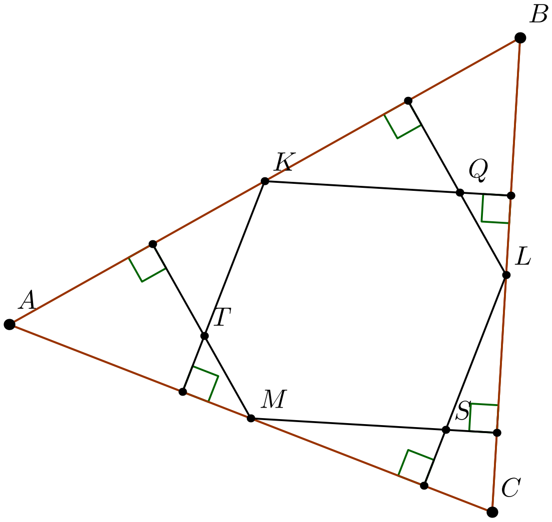
- Suppose the initial displayed number is 1. Give a sequence of exactly eight button presses that will result in a display of 313.
- Suppose the initial displayed number is 1, and we then perform exactly eight button presses. Describe all the numbers that can possibly result? Prove your answer by explaining how all these numbers can be produced and that no other numbers can be produced.

3 The *distinct prime factors* of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5.

Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$).
Show that, for every integer k greater than or equal to 2,

- A and B have the same set of distinct prime factors.
- $A + 1$ and $B + 1$ have the same set of distinct prime factors.

4 In an acute triangle ABC let $K, L,$ and M be the midpoints of sides $AB, BC,$ and $CA,$ respectively. From each of $K, L,$ and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and $CA,$ etc. The resulting 6 perpendiculars intersect at points $Q, S,$ and T as in the figure to form a hexagon $KQLSMT$ inside triangle $ABC.$ Prove that the area of this hexagon $KQLSMT$ is half of the area of the original triangle $ABC.$



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5 For $n > 1$ consider an $n \times n$ chessboard and place identical pieces at the centers of different squares.

- Show that no matter how $2n$ identical pieces are placed on the board, that one can always find 4 pieces among them that are the vertices of a parallelogram.
- Show that there is a way to place $(2n-1)$ identical chess pieces so that no 4 of them are the vertices of a parallelogram.

– BAMO-12

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1 Same as BAMO-8#3.

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2 Same as BAMO-8#4.

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3 Same as BAMO-8#5.

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4 Find a positive integer N and $a_1, a_2, \cdots, a_N$ where $a_k = 1$ or $a_k = -1$, for each $k = 1, 2, \cdots, N,$

such that
$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 \cdots + a_N \cdot N^3 = 20162016$$
or show that this is impossible.

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5 The corners of a fixed convex (but not necessarily regular) n-gon are labeled with distinct letters. If an observer stands at a point in the plane of the polygon, but outside the polygon, they see the letters in some order from left to right, and they spell a "word" (that is, a string of letters; it doesn't need to be a word in any language). For example, in the diagram below (where $n = 4$), an observer at point X would read "$BAMO$," while an observer at point Y would read "$MOAB$."

Diagram to be added soon

Determine, as a formula in terms of n, the maximum number of distinct n-letter words which may be read in this manner from a single n-gon. Do not count words in which some letter is missing because it is directly behind another letter from the viewer's position.

– `http://tinyurl.com/adihaya-bamo-notice` These problems were created by the BAMO Organizing Con

The Bay Area Mathematical Olympiad committee can be emailed at bamo@msri.org (`mailto:bamo@msri.org`). Visit their website at bamo.org (`http://bamo.org`).