Bulgaria National Olympiad 1996, Round 4

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Day 1

1 Find all prime numbers p, q for which pq divides $(5^p - 2^p)(5^q - 2^q)$.

2 Find the side length of the smallest equilateral triangle in which three discs of radii $2, 3, 4$ can be placed without overlap.

3 The quadratic polynomials f and g with real coefficients are such that if $g(x)$ is an integer for some $x > 0$, then so is $f(x)$. Prove that there exist integers m, n such that $f(x) = mg(x) + n$ for all x.

Day 2

1 Sequence $\{a_n\}$ it define $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}$$

for all $n \geq 1$

Prove that $\lfloor a_n^2 \rfloor = n$ for all $n \geq 4$.

2 The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet each other in the point E, while the diagonals AC and BD in the point F. The circumcircles of the triangles AFD and BFC have a second common point, which is denoted by H. Prove that $\angle EHF = 90^\circ$.

3 A square table of size 7×7 with the four corner squares deleted is given.

- What is the smallest number of squares which need to be colored black so that a $5-$square entirely uncolored Greek cross (Figure 1) cannot be found on the table?
- Prove that it is possible to write integers in each square in a way that the sum of the integers in each Greek cross is negative while the sum of all integers in the square table is positive.

Figure 1.