Round 4

www.artofproblemsolving.com/community/c2382128

by jasperE3

– Day 1

Problem 1 In triangle ABC, point O is the center of the excircle touching the side BC, while the other two excircles touch the sides AB and AC at points M and N respectively. A line through O perpendicular to MN intersects the line BC at P. Determine the ratio AB/AC, given that the ratio of the area of $\triangle ABC$ to the area of $\triangle MNP$ is $2R/r$, where R is the circumradius and r the inradius of $\triangle ABC$.

Problem 2 Prove that the sequence (a_n), where

$$a_n=\sum_{k=1}^n\left\{\frac{\left\lfloor2^{k-\frac12}\right\rfloor}2\right\}2^{1-k},$$

converges, and determine its limit as $n\to\infty$.

Problem 3 Let p be a real number and $f(x)=x^p-x+p$. Prove that:

(a) Every root α of $f(x)$ satisfies $|\alpha|<p^{\frac1{p-1}}$;
(b) If p is a prime number, then $f(x)$ cannot be written as the product of two non-constant polynomials with integer coefficients.

– Day 2

Problem 4 At each of the given n points on a circle, either $+1$ or -1 is written. The following operation is performed: between any two consecutive numbers on the circle their product is written, and the initial n numbers are deleted. Suppose that, for any initial arrangement of $+1$ and -1 on the circle, after finitely many operations all the numbers on the circle will be equal to $+1$. Prove that n is a power of two.

Problem 5 Prove that the perpendiculars, drawn from the midpoints of the edges of the base of a given tetrahedron to the opposite lateral edges, have a common point if and only if the circumcenter of the tetrahedron, the centroid of the base, and the top vertex of the tetrahedron are collinear.

Problem 6 Let x,y,z be pairwise coprime positive integers and $p\ge5$ and q be prime numbers which satisfy the following conditions:

(i) $6p$ does not divide $q-1$;
(ii) q divides x^2+xy+y^2;
(iii) q does not divide $x+y-z$.

Prove that $x^p+y^p \neq z^p$.